Virtual curve counting on Calabi-Yau 3-folds: A primer on DT/PT theory

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Contents

Introduction

Trying to understand our universe, string theorists came up with the model describing our world as a product of four-dimensional spacetime and a complex Calabi-Yau threefold. By a *Calabi-Yau threefold*, we here refer to a smooth, 3-dimensional projective variety X over $\mathbb{C}$, with trivial canonical bundle K_X and $H^1(X, \mathcal{O}_X) = 0$. Under this model, endpoints of strings will be attached to *D-branes*, which wrap around curves in the Calabi-Yau 3-fold. It has thus become a great interest, both for physicists and mathematicians, to try to understand the geometry of curves in Calabi-Yau (CY) 3-folds.

A distinctive property of CY 3-folds is their symmetric nature in cohomology, as a consequence of Serre duality and the fact that $K_X \cong \mathcal{O}_X$. A result of this is that the number of curves in a given topological class is expected to be finite, yielding an integer count which is expected to be invariant upon deforming the 3-fold.

There will be a scheme, called a *moduli space*, which parameterizes the curves in a certain class for a given CY 3-fold X. In degenerate cases, the number of objects in the moduli space might deviate from the number we normally would expect. In certain cases, the moduli space might even be of positive dimension. To come around this issue, we instead associate an invariant to each CY 3-fold which describes how many curves we would expect in a good setting. This number will agree with the actual count in good settings and will be invariant upon deforming into a possibly degenerate setting.

The reason why there is no canonical way of choosing such an invariant, is that the moduli spaces in question are not necessarily compact. We will in this essay examine two possible compactifications, namely the *Hilbert scheme*, parameterizing closed subschemes of X, and the space of *stable pairs*, parameterizing certain pairs $(\mathcal{F}, s)$ of a sheaf equipped with a section. We will explore the methods of assigning virtual counting invariants to these moduli spaces, and see how the invariants relate.

The invariants associated to the compactification via the Hilbert scheme are known as *Donaldson-Thomas* (or *DT*) *invariants*, and were introduced in 1998 by Thomas in the paper [**Th00a**]. The paper deals with the more general theory of counting *stable coherent sheaves*, but we will focus on the 1-dimensional case here. Later, in 2007, Pandharipande and Thomas came up with alternative invariants, known as *PT invariants*, associated to the moduli space of stable pairs (see [**PT09**]). The purpose was to reduce the overcount caused by the presence of additional 0-dimensional objects in the Hilbert scheme.

There are many other contributors to this substantial area, including Behrend-Fantechi [**BF97**], Li-Tian [**LT98**], Bridgeland [**Br11**], Toda [**To14**] and Kontsevich-Soibelman [**KS13**]. There is also a different side of this theory, which I yet have to discover, arising from symplectic topology.

Upon choosing essay topic, I looked for something wondrous and unfamiliar to explore. Having glanced through the paper [**Th00a**] by Thomas, I knew that I would have made a grand accomplishment once I could grasp these concepts. Indeed, I have been fully captivated from start to finish, alternating between states of enlightenment and puzzlement. I have had the great pleasure to discuss with my advisor, Mark Gross, who enthusiastically answered a wide range of questions. After a vivid first essay meeting, Mark eventually concluded: *"I am afraid you are going off in too many directions!"*

I decided put the spotlight on two curve counting invariants, the DT- resp. PT invariants, but we will move around the scene throughout the play.

In Chapter 1, we introduce the concept of a moduli space and illustrate how compactifications can differ. In Chapter 2, we demonstrate the concept of virtual counting and explore the relations to intersection theory and deformation theory. In Chapter 3, we develop deformation theory to study the local nature of our moduli spaces. In Chapter 4, we describe the DT invariants. In Chapter 5, we acquaint ourselves with the derived category of coherent sheaves, which will be used for investigating the stable pairs, which are introduced in Chapter 6. In Chapter 7, we relate the DT- and PT invariants using identities in the motivic Hall algebra. The essay is concluded with a brief outlook on future research directions.

Lastly, I would like to thank Ashesh Bati, Patrick Luo, Radosłav Žak and Sae Koyama for the joyful AG-seminars we held for each other throughout Part III of the Mathematical Tripos. I also thank my lecturers in various courses in algebraic geometry; Dhruv Ranganathan, Renata Picciotto and Tony Scholl, whose passions for the subject made their courses presented in the best possible way.

CHAPTER 1

Moduli spaces

> *Each animal family keeps to itself,*
> *although it is understood that on a day*
> *like this universal peace reigns, and*
> *none need fear attack.*
>
> S. Lagerlöf, *The Wonderful Adventures of Nils*

By a *variety*, we here mean an integral, separated scheme of finite type over a field. A *curve* is a variety of dimension 1.

Studying curves in a projective variety X over $\mathbb{C}$, there are many questions that seem natural to ask. For instance, we might wonder how the curves intersect, or how we can deform them inside X. As these examples indicate, we are usually not only interested in the individual curves themselves, but also in their relation to other curves in X. It would therefore be convenient to be able to study several curves at the same time. Since there will be too many curves to handle if we do not impose further restrictions, we will mostly consider curves within a certain topological class $\beta \in H_2(X, \mathbb{Z})$. The subscript 2 comes from the fact that a complex curve C has 2 real dimensions, thus defining a class in the 2^{nd} homology by the inclusion $C \hookrightarrow X$ as real topological spaces. Moreover, we can restrict ourselves to curves of a given Euler characteristic $\chi(\mathcal{O}_C) = n \in \mathbb{Z}$.

Ideally, we would like to have a scheme $\mathcal{M}$ parameterizing these curves, which also describes how the curves vary in families. This leads us to the notion of a *fine moduli space.*

Definition 1.1. Let $h\colon \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sets}$ be a functor. In words, h associates to each complex scheme or *base* B a set $h(B)$ of *families* over B, which can be pulled back appropriately under a base change $B' \longrightarrow B$. A *fine moduli space* for h is a scheme $\mathcal{M}$ equipped with a *universal family* $U \in h(\mathcal{M})$, such that each family $\mathfrak{X}$ over B (up to suitable equivalence) can be obtained as a pullback of U along a unique morphism $B \xrightarrow{f_{\mathfrak{X}}} \mathcal{M}$.

Families $h(B)$ of interest can be, for instance, morphisms $\mathfrak{X} \xrightarrow{f} B$ with a specified class of fibers, or certain sheaves on $X \times B$ for a fixed variety X. The existence of a fine moduli space $\mathcal{M}$ cannot be guaranteed in general, but is something worth striving for.

If we want to study the curves in a fixed class (β, n) on a complex variety X, we might be interested in families of subschemes $Y \subset X \times B$ for various bases B, whose restriction to a fixed $\mathbb{C}$-point is a curve in the class (β, n). Moreover, it is reasonable to include the assumption that Y is flat over B, so that the fibers do not vary uncontrollably. Assuming that there is a fine moduli space $\mathcal{M}(X, \beta, n)$ for these families, a $\mathbb{C}$-point of $\mathcal{M}(X, \beta, n)$ corresponds precisely to a curve C in the class (β, n). In this way, a fine moduli space both parameterizes the objects we are interested in and captures how they vary in families.

Even though we might be able to find a fine moduli space $\mathcal{M}(X, \beta, n)$ for the curves in a fixed class (β, n), this space need not be compact. Compactness would make it easier to study $\mathcal{M}$, e.g. by intersection theory (see Chapter 2). We are therefore led to consider possible compactifications, and hope that they also can be obtained as fine moduli spaces.

We will here study curves in a class (β, n) in a Calabi-Yau 3-fold X. We will examine two possible compactifications of the moduli space $\mathcal{M}(X, \beta, n)$ and compare their properties.

The first compactification will be the Hilbert scheme $I_n(X, \beta)$, which is a fine moduli space for closed subschemes C in X of dimension ≤ 1, with $\chi(\mathcal{O}_C) = n$ and $[C] = \beta \in H_2(X, \mathbb{Z})$. Note that this can include combinations of components of dimensions 0 and 1, as well as nonreduced subschemes, apart from the curves in the class (β, n).

The other compactification $P_n(X, \beta)$ is a fine moduli space parameterizing *stable pairs*. A stable pair consists of pair $(\mathcal{F}, s)$ of a coherent sheaf $\mathcal{F}$ on X, with support class $[\mathrm{Supp}(\mathcal{F})] = \beta \in H_2(X, \mathbb{Z})$ and Euler characteristic $\chi(\mathcal{F}) = n$, and a section $s \in H^0(X, \mathcal{F})$, with the following properties:

- The support of $\mathcal{F}$ is 1-dimensional.
- $\mathcal{F}$ is *pure*, meaning that subsheaves $0 \neq \mathcal{G} \subset \mathcal{F}$ satisfy $\dim(\mathrm{Supp}(\mathcal{G})) = 1$.
- The section s, viewed as a map $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ by sending $1 \in \mathcal{O}_X(U)$ to $s|_U \in \mathcal{F}(U)$ for each open set $U \subset X$, has $\dim(\mathrm{coker}(s)) = 0$.

Geometrically, a stable pair should be thought of as a pure 1-dimensional closed subscheme $C = \mathrm{Supp}(\mathcal{F})$ with no embedded points, together with the information of additional 0-dimensional points located on C, arising from $\mathrm{Supp}(\mathrm{coker}(s))$.

Note that a curve C corresponds to the pure sheaf $\mathcal{O}_C$, equipped with the section 1 which has empty cokernel. Therefore, these moduli spaces parameterize, among other things, all the curves in a class (β, n). Moreover, both $I_n(X, \beta)$ and $P_n(X, \beta)$ are known to be proper.

We will illustrate how $I_n(X, \beta)$ resp. $P_n(X, \beta)$ can be seen as compactifications of the moduli space of curves in a class (β, n).

Example 1.2. Consider the flat family of rational curves $C_a \in \mathbb{P}^3$, for $a \in \mathbb{C}^*$, given in an affine patch by the parameterizations $\iota_a \colon \mathbb{A}_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ sending

$$t \mapsto (t^2 - 1, t^3 - t, at)$$

(from [**Ha77**, III.9.8.4]). The ideal sheaf of a curve C_a in this patch is given by

$$I_a = (a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)).$$

Considering the limiting ideal sheaf as $a \to 0$ in $\mathbb{C}$, we obtain a limiting element in the Hilbert scheme $I_1(\mathbb{P}^3, [\mathbb{P}^1])$ given by the ideal sheaf

$$I_0 = (z^2, yz, xz, y^2 - x^2(x+1)).$$

This corresponds to an 1-dimensional subscheme consisting of a nodal cubic in the xy-plane, together with an embedded point at the origin which remembers the z-direction along which the deformation took place (see Figure 1).

On the other hand, the structure sheaf of this limiting subscheme does not define an element of $P_1(\mathbb{P}^3, [\mathbb{P}^1])$, because, as we will see in Chapter 6, the (scheme theoretic) support of a pure sheaf cannot contain embedded points. Instead, we construct the stable pair limit of the above family as follows: Consider the family of maps

$$\begin{aligned} s_a \colon \mathbb{C}[x,y,z] &\to \mathbb{C}[t] \\ x &\mapsto t^2 - 1 \\ y &\mapsto t^3 - t \\ z &\mapsto at, \end{aligned}$$

where $a \in \mathbb{C}^*$. These should be viewed as stable pairs (F_a, s_a) of $\mathbb{A}^3_{\mathbb{C}}$, where in this case $F_a = \mathbb{C}[t]$ for all $a \in \mathbb{C}^*$.

The limiting stable pair (F_0, s_0) is given by the map

$$\begin{aligned} s_0 \colon \mathbb{C}[x,y,z] &\to \mathbb{C}[t] \\ x &\mapsto t^2 - 1 \\ y &\mapsto t^3 - t \\ z &\mapsto 0, \end{aligned}$$

which is not surjective as t does not lie in the image of s_0. In other words, the curve $\operatorname{Spec}(\mathbb{C}[t])$ is no longer embedded at the origin. The support of F_0 is the nodal cubic $y^2 = x(x^2+1)$. The cokernel $\mathbb{C}\langle t\rangle / \mathbb{C}\langle t^2, t^3 - t\rangle$ is supported at the origin in $\mathbb{A}^3_{\mathbb{C}}$. In Chapter 6, we will see how these properties generalize to stable pairs in general, and what advantages they lead to.

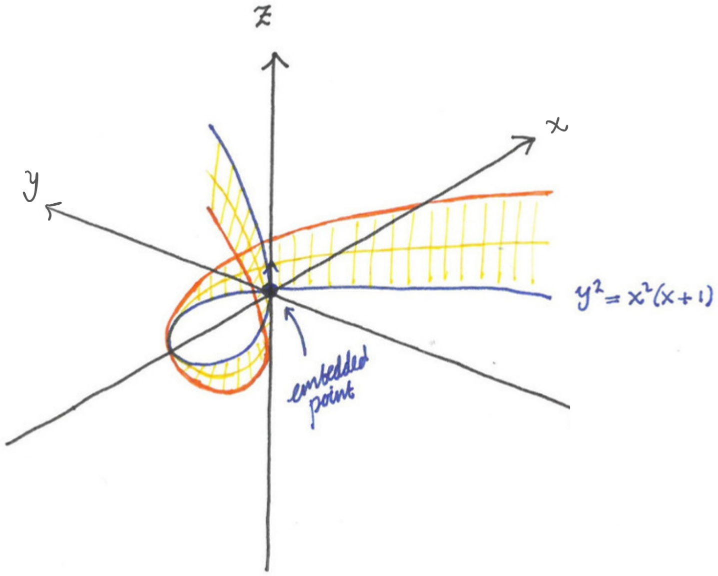

FIGURE 1. Taking a limit in the Hilbert scheme.

In the next chapter, we will go back to the general problem of counting objects in a moduli space, which for our purposes later will be useful for defining counting invariants associated to $I_n(X, \beta)$ and $P_n(X, \beta)$.

CHAPTER 2

Virtual counting

> *He tried Counting Sheep, which is sometimes a good way of getting to sleep, and, as that was no good, he tried counting Heffalumps.*
>
> A. A. Milne, *Winnie-The-Pooh*

When counting objects in algebraic geometry we often make use of intersection theory, which allows us to understand how closed subvarieties intersect in a given scheme.

A classical example is the use of Weil divisors. A Weil divisor on a normal variety is a formal integer combination of closed subvarieties of codimension 1. A version of Bezout's theorem gives us the number of intersections, with multiplicity, between two distinct irreducible plane curves C, D of degrees d_1 resp. d_2. The desired number d_1d_2 is obtained by intersecting the Weil divisors $[C]$, $[D]$ associated to the two curves in the Weil divisor class group. This reduces to finding the intersection number between the linearly equivalent divisors $d_1[L]$ resp. $d_2[L]$, where $[L]$ is the divisor associated to a line L. A useful property of the intersection number $[C].[D]$ is that it is defined, and integer valued, even when the curves C and D coincide. Whereas the number $[C].[D]$ only counts the number of (distinct) intersection points between $[C]$ and $[D]$ when C, D meet transversely, the ability to define this number in general still plays an important role in the nontransverse case. For instance, the number $[C].[D]$ here counts the number of intersection points between two curves whose corresponding divisors are linearly equivalent to $[C]$ and $[D]$, respectively, and whose intersection is transverse.

We can generalize this philosophy to many other counting problems, by a method of so-called "virtual counting". This amounts to finding a suitable invariant which counts the actual number in a nice (e.g. smooth, transverse) setting, whereas it might give other useful information in less well-behaved situations. For instance, rather than counting objects in a particular case, it is common to find interest in the behavior of counts when varying the setting. If, say, we are interested in counting the number of curves in a certain class in a given variety X, we might want to see how the number changes when varying X in a family. It might happen that a family contain singular fibers, for which the count deviates from the smooth case. Not only might we find an unusual number, e.g. due to nonreduced points or other singularities in the chosen moduli space containing our curves, but we might also find that the moduli space has a different dimension. This would make it hard to even count the parameterized objects properly. Instead, it is often better to associate a "virtual invariant", rather than the actual count, to X, which remains an integer and behaves well under deformations.

Counting objects is commonly done by intersecting classes associated to closed subvarieties of some scheme X in the *Chow group*, denoted $\mathcal{A}_*(X)$. This group generalizes the group of Weil divisors to include subvarieties of arbitrary codimension. Each closed subvariety Y of X corresponds to a class (or *cycle*) $[Y]$ in $\mathcal{A}_*(X)$, which can be pulled back to the intrinsic class $[Y]$ in $\mathcal{A}_*(Y)$ under the inclusion map $\iota \colon Y \hookrightarrow X$. The class $[Y]$ in $\mathcal{A}_*(Y)$ is called the *fundamental class* of Y.

If Y is the moduli space we are interested in, we can sometimes choose a more suitable class in $\mathcal{A}_*(Y)$, called a *virtual fundamental class*, from which we can obtain the virtual count we are looking for. As before, we hope that this class agrees with the usual fundamental class in nondegenerate settings, whereas it might have deformation-invariant properties that also apply to degenerate situations.

2.1. A vector bundle scenario

A well-understood case takes the shape of a rank r vector bundle

$$\begin{tikzcd}
E \arrow[d, "\pi"'] \\
Z
\end{tikzcd}$$

over an n-dimensional nonsingular variety Z. Assume that we have a section $s \colon Z \hookrightarrow E$ whose intersection with the zero section gives a moduli space $\mathcal{M}$, for which we wish to assign a virtual fundamental class.

The "good" situation here is when s is *transverse* to the zero section Z_0, i.e. when $\operatorname{codim}_E(\mathcal{M}) = \operatorname{codim}_E(Z_0) + \operatorname{codim}_E(s(Z))$. Since $\dim(E) = n+r$, we obtain $(n+r) - \dim(\mathcal{M}) = r+r$, i.e. $\dim(\mathcal{M}) = n-r$. In this case, the virtual fundamental class $[\mathcal{M}]^{vir}$, lying inside $\mathcal{A}_*(\mathcal{M})$, is just the fundamental class $[M]$.

Tools of intersection theory will allow us to deal with the nontransverse case or cases where there are no nonzero sections to begin with. By considering the sections λs for $\lambda \in \mathbb{A}_1$ and taking the limit $\lambda \to \infty \in \mathbb{P}_1$, we obtain a *normal cone* $C_{\mathcal{M}/Z}$ of the restricted bundle $E|_{\mathcal{M}}$. This subscheme corresponds to a cycle of the Chow group $\mathcal{A}_*(E|_{\mathcal{M}})$, by taking a suitable linear combination of the irreducible components of $(C_{\mathcal{M}/Z})_{red}$. We can relate this cycle in $\mathcal{A}_*(E|_{\mathcal{M}})$ to one in $\mathcal{A}_*(\mathcal{M})$, by a useful homomorphism between these groups which we are soon to encounter. By deformation invariant properties of the intersection product for Chow groups, this cycle will be an appropriate virtual fundamental class for the zero section of s.

For the sake of completeness, we first give a clear definition of Chow groups.

Definition 2.1. For a scheme X, the group $Z_k(X)$ of *k-cycles* consists of formal integer combinations $\sum n_i[V_i]$, for closed k-dimensional subvarieties V_i of X.

A nonzero function f in the function field of any $(k+1)$-dimensional subvariety W determines a k-cycle: Indeed, we can define $[\operatorname{div}(f)] := \sum \operatorname{ord}_V(f)[V]$ for k-dimensional subvarieties V of W.

The group $A_k(X)$ is defined to be $Z_k(X)$ modulo the following equivalence relation: Elements of the group generated by cycles $[\operatorname{div}(f_i)]$, for f_i a nonzero function in the function field of a $(k+1)$-dimensional subvariety V_i, are equivalent to 0. The *Chow group* $A_*(X)$ is the graded sum $\bigoplus_{k=0}^{\dim(X)} A_k(X)$, and its elements are called *cycles*.

The abovementioned equivalence relation is referred to as *rational equivalence*, and has the following geometric interpretation: Consider two cycles C, D in the chow group $A_*(X)$ associated to a variety X. If there is a family of cycles on X parameterized by $\mathbb{P}_1$, i.e. a cycle in $A_*(X \times \mathbb{P}_1)$, such that the restrictions to $X \times \{0\}$ and $X \times \{\infty\}$ give the cycles C resp. D, then C and D are rationally equivalent.

Definition 2.2. A closed subscheme Y of a scheme X can be associated with a cycle, denoted $[Y]$, in $A_*(X)$ as follows: We define $[Y]$ to be the sum of the cycles of the irreducible components of Y_{red} with suitable geometric multiplicities (see [**EH16**, p.15]). The *fundamental class* of a scheme X is the cycle $[X] \in A_*(X)$.

A flat morphism $f\colon X \longrightarrow Y$ (of relative dimension r) of schemes induces a functorial pullback map $f^*\colon A_{k-r}(Y) \longrightarrow A_k(X)$, by letting $f^*[V] := [f^{-1}(V)]$.

A key result for vector bundles $E \xrightarrow{\pi} Z$ of rank r, where Z has dimension n, is that the pullback map $\pi^*\colon A_{k-r}(Z) \longrightarrow A_k(E)$ is an isomorphism ([**Fu98**, p.64]). The inverse map $0^!_E\colon A_k(E) \longrightarrow A_{k-r}(X)$ is called the *Gysin homomorphism.* Going back to our vector bundle $E|_{\mathcal{M}} \longrightarrow \mathcal{M}$, we obtain a map $0^!_{E|_{\mathcal{M}}}\colon A_n(E|_{\mathcal{M}}) \longrightarrow A_{n-r}(\mathcal{M})$, and hence a class $0^!_{E|_{\mathcal{M}}}([C_{\mathcal{M}/Z}])$ in $A_{n-r}(\mathcal{M})$. We define the virtual class $[\mathcal{M}]^{\text{vir}}$ to be $0^!_{E|_{\mathcal{M}}}([C_{\mathcal{M}/Z}])$, which has the dimension $n-r$. Note that this is the dimension we would expect in a transverse setting, and is therefore suitably referred to as the *virtual dimension* of $\mathcal{M}$, denoted $\text{vdim}(\mathcal{M})$.

Until now, we assumed that $\mathcal{M}$ belongs to an ambient space Z with a section $Z \overset{s}{\hookrightarrow} E$ whose intersection with the zero section cuts out $\mathcal{M}$.

In accordance with the local nature of the problem, it turns out that the data $TZ|_{\mathcal{M}} \xrightarrow{ds} E|_{\mathcal{M}}$ is sufficient for the construction of a virtual fundamental class.

Our geometric intuition tells us that $\ker(ds)$ is $T\mathcal{M}$. We will verify this explicitly.

Claim 2.3. In the above setting, we have an exact sequence:

$$0 \longrightarrow T\mathcal{M} \longrightarrow TZ|_{\mathcal{M}} \xrightarrow{ds} E|_{\mathcal{M}} \xrightarrow{\pi_s} \text{Ob} \longrightarrow 0 \tag{1}$$

for some cokernel Ob.

Proof. We will find an exact sequence

$$E^\vee|_{\mathcal{M}} \longrightarrow \Omega_Z|_{\mathcal{M}} \longrightarrow \Omega_{\mathcal{M}} \longrightarrow 0 \tag{2}$$

of cotangent sheaves, which we then dualize to give the sequence (1).

Let $U = \text{Spec}(B) \subset Z$ be an affine open such that $\mathcal{M}|_U = \text{Spec}(B/I)$, for some ideal $I \subset B$, and $E|_U = \text{Spec}(B[x_1, \ldots, x_r])$.

The fiber diagram

$$\begin{tikzcd}
\mathcal{M}|_U \arrow[r, hook, "j"] \arrow[d, hook, "j"] & U \arrow[d, hook, "s"] \\
U \arrow[r, hook, "i"] & E|_U
\end{tikzcd}$$

corresponds to the maps of B-algebras

$$\begin{tikzcd}
B/I & B \arrow[l, hook', "j^{\#}"'] \\
B \arrow[u, "j^{\#}"] & B[x_1, \dots, x_r], \arrow[l, hook', "i^{\#}"'] \arrow[u, "s^{\#}"']
\end{tikzcd} \tag{3}$$

where i comes from the inclusion map of the zero section, and j from the inclusion of $\mathcal{M}$ into Z.

Since equation (3) comes from a fiber diagram, where $i^{\#}$ maps the variables x_i to 0 and $j^{\#}$ is the projection map, the ideal in B generated by the elements $s^{\#}(x_i)$ is precisely the ideal I.

The embedding $\mathcal{M}|_U \xhookrightarrow{j} U$ induces a surjective map from $\Omega_U|_{\mathcal{M}|_U}$ to $\Omega_{\mathcal{M}|_U}$ given by $db \mapsto d(b+I)$, with kernel precisely dI. Moreover, we have a map from $E^\vee|_{\mathcal{M}|_U} = B/I\langle dx_1, \dots, dx_r\rangle$ to $\Omega_U|_{\mathcal{M}|_U}$ sending each dx_i to $d(s^{\#}(x_i))$, thus having image dI.

We consequently obtain an exact sequence

$$E^\vee|_{\mathcal{M}|_U} \longrightarrow \Omega_U|_{\mathcal{M}|_U} \longrightarrow \Omega_{\mathcal{M}|_U} \longrightarrow 0.$$

By considering a cover of Z by open affines U with $E|_U$ free, we obtain the desired exact sequence (2). ☐

From the sequence (1), we also see that $\operatorname{vdim}(\mathcal{M}) = \dim(Z) - \operatorname{rank}(E) = \operatorname{rank}(TZ|_{\mathcal{M}}) - \operatorname{rank}(E|_{\mathcal{M}})$.

Assume that $\mathcal{M}$ parameterizes certain objects associated to a fixed variety X. The above sequence then arises from the deformation theory of the objects parameterized by $\mathcal{M}$. The tangent space $T_p\mathcal{M}$ represents the first order deformations of an object p inside X, while Ob_p measures the obstructions for extending these deformations to higher order.

A motivation for the obstruction sheaf being the cokernel of the sequence is as follows: The cokernel Ob_p measures the failure of the map ds to be surjective at p. Surjectivity of ds means that s is transverse to the zero section at p. We can make an analogy with the transversality theorem for manifolds, which states that, for a smooth map $f\colon X \longrightarrow Y$ between manifolds, and $N \subset Y$ a submanifold which is transverse to f, the preimage $f^{-1}(N)$ is a submanifold of X. This "smoothness" of the preimage can be compared with "higher order neighborhoods" of p inside $\mathcal{M}$.

In greater generality, when our moduli space is fine, we can define a virtual fundamental class from the following data:

$$0 \longrightarrow T\mathcal{M} \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \mathrm{Ob} \longrightarrow 0 \tag{4}$$

where E_1, E_2 are locally free sheaves satisfying the conditions for a *perfect obstruction theory*, which will not be discussed here. The reader is referred to [**Ri22**, p.170].

From the sequence (4), we can define an analogous cone $C_{\mathcal{M}} \subset E_1$, which we can pull back to a class on $\mathcal{M}$ using the Gysin homomorphism to obtain a virtual fundamental class (see [**BF97**]). The virtual class will be of dimension $\mathrm{vdim}(\mathcal{M}) = \dim(E_1) - \dim(E_2)$.

In the coming chapter, we will be concerned with moduli spaces of coherent sheaves. Using deformation theory, we will find $T\mathcal{M}$ and Ob explicitly in this setting. This will allow us to define a virtual fundamental class for each of the moduli spaces $I_n(X, \beta)$ and $P_n(X, \beta)$ associated to a Calabi-Yau 3-fold X. Since these moduli spaces have virtual dimension 0, we will be able to take the degree of these classes to obtain the integer-valued DT invariants and PT invariants, respectively.

You should keep the sequence (4) in mind to remember the roles of the constituent terms that we will explicitly find in the next chapter, using deformation theory.

CHAPTER 3

Deformation theory

> *The wild geese were never so merry as when flying over a flat country.*
>
> S. Lagerlöf, *The Wonderful Adventures of Nils*

What does it mean to deform an algebro-geometric object P, such as a scheme, sheaf or a morphism? The intuitive idea will be to think of a family of objects, varying continuously over a base, with P corresponding to a particular closed fiber.

For instance, given a scheme X over k, a deformation of a closed subscheme $Y \subset X$ is a closed subscheme $Y' \subset X \times A$ over a base A such that $Y' \times a \xrightarrow{\sim} Y$ for a specified k-point $a \in A$. Moreover, we require that the morphism $Y' \xrightarrow{\pi} A$ induced by the projection from $X \times A$ to A is flat. This condition ensures that the fibers vary nicely in the family. As an example, the dimension of a fiber will be a local invariant under a flat morphism.

Deformations can be difficult to study globally, since global deformations seldom exist. Hence, we often restrict to local settings by considering very small extensions of the base. For instance, if X is a scheme over the base field k, we can consider the thickened ring $D = k[\epsilon]/(\epsilon^2)$. The ring D is the *ring of dual numbers*, and the corresponding scheme $\mathrm{Spec}(D)$ has the same underlying topological space as $\mathrm{Spec}(k)$, i.e. a point. The difference is that the thickened ring is equipped with an extra variable ϵ, whose square is 0. This variable might be treated as an infinitesimal scalar which will allow us to make small perturbations. A deformation over D is referred to a *first order deformation*.

As an example, let us consider first order deformations of the cuspidal cubic $Y = \mathrm{V}(y^2 - x^3)$ inside $X = \mathrm{Spec}(k[x, y])$. This corresponds to choosing an ideal $I' \subset k[x, y, \epsilon]/(\epsilon^2)$ such that I' restricts to $I = (y^2 - x^3)$ modulo ϵ. For example, we can pick $I' = (y^2 - x^3 + \epsilon x^2)$. Pictorially, we are deforming the embedded cuspidal cubic "infinitesimally" into an embedded nodal cubic.

Deforming objects parameterized by a moduli space $\mathcal{M}$ will take us to nearby points in the moduli space. In particular, specifying a deformation of an object up to first order will be the same as choosing a tangent direction at the corresponding point in the moduli space. This is why deformations over the dual numbers will be of interest when studying the local structure of a moduli space. For instance, let X be a scheme over k, and consider a fine moduli space $\mathcal{M}$ associated to certain objects of X (e.g. subschemes, sheaves etc.). To deform an object P corresponding to a point $p \in \mathcal{M}$ means associating a similar object P' to $X \times B$ over a base scheme B, so that the restriction of P' to a specified closed point recovers P (with appropriate flatness assumptions). In the language of Definition 1.1, we say that P' is a *family* over the base B.

To give a family P' over the dual numbers D corresponds to giving a classifying morphism $\operatorname{Spec}(D) \longrightarrow \mathcal{M}$, by the definition of a fine moduli space. Morphisms from the dual numbers to the scheme $\mathcal{M}$, with image a specified closed point p, correspond to elements in $T_p\mathcal{M} := \operatorname{Hom}(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$ where $\mathfrak{m}_p \in \mathcal{O}_{X,p}$ is the maximal ideal.

Having considered first order deformations, we might ask whether it is possible to extend these deformations to higher order, e.g. to bases $k[\epsilon]/(\epsilon^n)$ for $n \geq 2$. In the example with the embedded cuspidal cubic, we are free to add any multiple of ϵ^n to the ideal. For general closed subschemes, it might not only be difficult to extend deformations in affine patches, if they exist, to higher order, but we might also encounter problems when trying to glue these deformations together.

When considering higher order deformations, e.g. over $k[\epsilon]/(\epsilon^n)$ for $n \geq 2$, it might be cumbersome to deal with all the terms $1, \epsilon, \ldots, \epsilon^{n-1}$ simultaneosly. Instead, having deformed an object to first order, we will try to extend it to higher orders one step at a time. Given a deformation of the object over a base scheme B, we try to further extend the deformation over a “thicker” base B'. By thicker, we refer to the idea that the topological space remains the same, while some additional “higher order information” is encoded in B'. Formally, we consider a closed embedding $B \subset B'$ defined by a nilpotent ideal $\mathcal{I}$, i.e. $\mathcal{I}^n = 0$ for some $n \geq 2$. We can consider smaller extensions one at a time, noting that $\mathcal{I}^{n-1}$ squares to 0, by restricting to the case $n = 2$. An example to keep in mind is the choice of bases $B = k[\epsilon]/(\epsilon^{n-1})$, $B' = k[\epsilon]/(\epsilon^n)$ for some $n \geq 2$.

3.1. Deforming coherent sheaves

Deforming a subscheme $Y \subset X$ over the field k can be seen as deforming the ideal sheaf $\mathcal{I}_Y$ associated to Y. More generally, we can deform any coherent sheaf $\mathcal{F}$ on X. Given a coherent sheaf $\mathcal{F}$ on X, a *deformation* of $\mathcal{F}$ over a base B is a sheaf $\mathcal{F}'$ on $X \times B$, flat over B, equipped with a fixed isomorphism $\mathcal{F}'_b \xrightarrow{\sim} \mathcal{F}$ where $\mathcal{F}'_b$ is the restriction of $\mathcal{F}'$ over some k-point $b \in B$.

Consider an extension $B \subset B'$ of the base defined by an ideal sheaf $\mathcal{I}$ of B' whose square is 0. If $\mathcal{E}$ is a coherent sheaf on X and $\mathcal{F}$ is a deformation of $\mathcal{E}$ over B, an *extension* of $\mathcal{F}$ over B' is a coherent sheaf $\mathcal{F}'$ on $X \times B'$ such that the restriction $\mathcal{F}' \otimes_{\mathcal{O}_{X\times B'}} \mathcal{O}_{X\times B} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism. We call the deformation $\mathcal{F}$ *obstructed* if it cannot be extended over B'.

In good cases, the deformations and obstructions can be classified by explicit groups. We will illustrate this for line bundles on a scheme X over k.

THEOREM 3.1. *Let X be a scheme over k, $\mathcal{L}_0$ a line bundle on X, and $Y_n = \operatorname{Spec}(k[\epsilon]/(\epsilon)^n)$.*

1. The deformations $\mathcal{L}$ of $\mathcal{L}_0$ over Y_1 are in correspondence with the elements in $\mathrm{H}^1(X, O_X)$.

2. Let $\mathcal{L}_n$ be a deformation of $\mathcal{L}_0$ over Y_n. Then $\mathcal{L}_n$ can be identified with an element δ in $\mathrm{H}^2(X, O_X)$, where $\delta = 0$ if and only if $\mathcal{L}_n$ can be extended to a line bundle $\mathcal{L}_{n+1}$ over Y_{n+1}.

PROOF. 1. See e.g. [**Ha10**, p.13]. We will instead prove a general statement for coherent sheaves in Theorem 3.2.

2. Cover X by open sets U_i such that $\mathcal{L}_n|_{U_i \times Y_n}$ is trivial. The line bundle $\mathcal{L}_n$ is then determined by its transition functions on $U_{ij} := U_i \cap U_j$, viewed as $\mathcal{O}_{U_{ij}\times Y_n}$-linear isomorphisms $\varphi_{ij} \colon \mathcal{O}_{U_{ij}\times Y_n} \xrightarrow{\sim} \mathcal{O}_{U_{ij}\times Y_n}$. Moreover, we have the gluing relations $\varphi_{ij} = \varphi_{ji}^{-1}$, $\varphi_{ii} = \mathrm{id}$, and on triple overlaps $U_{ij} \cap U_{jk} \cap U_{ik}$ the cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = \mathrm{id}$. An isomorphism φ_{ij} is just a multiplication by a scalar in $\mathcal{O}^*_{U_{ij}\times Y_n}$, and can hence be seen as an element $\varphi_{ij} = \sum_{k=0}^{n-1} \varphi_{ijk}\epsilon^k \mod \epsilon^n$ with $\varphi_{ijk} \in \mathcal{O}_{U_{ij}}$ and $\varphi_{ij0} \in \mathcal{O}^*_{U_{ij}}$. In order to find an extension $\mathcal{L}_{n+1}$ of $\mathcal{L}_n$ to Y_{n+1}, we first start by picking trivial sheaves $\mathcal{L}^i_{n+1}$ on the patches $U_i \times Y_{n+1}$. These extend the sheaves $\mathcal{L}_n|_{U_i\times Y_n}$ by the isomorphism $\mathcal{O}_{U_{ij}\times Y_{n+1}} \otimes_{\mathcal{O}_{U_{ij}\times Y_{n+1}}} \mathcal{O}_{U_{ij}\times Y_n} \xrightarrow{\sim} \mathcal{O}_{U_{ij}\times Y_n}$ arising from the closed embedding $Y_n \hookrightarrow Y_{n+1}$.

In order to glue these local extensions together, we need to pick transition functions $\tilde{\varphi}_{ij} \colon \mathcal{O}_{U_{ij}\times Y_{n+1}} \xrightarrow{\sim} \mathcal{O}_{U_{ij}\times Y_{n+1}}$ extending the φ_{ij}'s. We pick arbitrary lifts $\tilde{\varphi}_{ij} = \sum_{k=0}^{n-1} \varphi_{ijk}\epsilon^k + a_{ij}\epsilon^n \mod \epsilon^{n+1}$ of the φ_{ij}'s to $\mathcal{O}_{U_{ij}\times Y_{n+1}}$ where $a_{ij} \in \mathcal{O}_{U_{ij}}$, such that $\tilde{\varphi}_{ij} = \tilde{\varphi}_{ji}^{-1}$ and $\tilde{\varphi}_{ii} = \mathrm{id}$.

Since the φ_{ij}'s together determine the global line bundle $\mathcal{L}_n$, they satisfy the cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1 \mod \epsilon^n$. Thus, the lifts $\tilde{\varphi}_{ij}$ satisfy the relation $\tilde{\varphi}_{ij}\tilde{\varphi}_{jk}\tilde{\varphi}_{ki} = 1 + b_{ijk}\epsilon^n \mod \epsilon^{n+1}$ for some $b_{ijk} \in \mathcal{O}_{U_{ij}}$. Changing the a_{ij}'s to $a_{ij} + a'_{ij}$, the coefficient b_{ijk} is replaced by $b_{ijk} + (a'_{ij} + a'_{jk} - a'_{ik})$. In other words, the coefficients b_{ijk} have changed by a Čech 1-cocycle for the sheaf $\mathcal{O}_X$ with respect to the cover U_{ij} of X. Thus, the elements b_{ijk} form a well-defined 2-cocycle δ in $H^2(X, \mathcal{O}_X)$. The element δ will vanish precisely when we can choose the a_{ij}'s such that $\tilde{\varphi}_{ij}\tilde{\varphi}_{jk}\tilde{\varphi}_{ki} = 1 \mod \epsilon^{n+1}$. This precisely means that the sheaves $\mathcal{L}^i_{n+1}$ can be glued together to a global extension $\mathcal{L}_{n+1}$ of $\mathcal{L}_n$. Thus, the group $H^2(X, \mathcal{O}_X)$ can be seen as an obstruction space for the possibility of extending deformations of line bundles on $X \times Y_n$ to line bundles on $X \times Y_{n+1}$. □

For our purposes, we will be interested in deforming coherent sheaves in general, since they include the ideal sheaves for the Hilbert scheme and the pure sheaves for the moduli space of stable pairs.

We consider a projective variety X over k, and a coherent sheaf $\mathcal{E}$ on X.

We start by considering the base extension $\mathrm{Spec}(k) \subset \mathrm{Spec}(D)$ for $D = k[\epsilon]/(\epsilon^2)$, defined by the ideal (ϵ) which squares to 0. We want to classify the extensions of $\mathcal{E}$ over $X \times D$ (with abuse of notation), as this will help us finding the tangent space to our moduli spaces $\mathcal{M}$ of certain sheaves of interest. More precisely, we will show the following theorem.

Theorem 3.2. *Let X be a scheme over a field k. Deformations $\mathcal{F}$ of a coherent sheaf $\mathcal{E}$ over $X \times D$ are in 1-1 correspondence with elements in the group* $\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})$.

Remark 3.3. How should we think about the Ext-groups here? A good way to think about the group $\mathrm{Ext}^1_R(A, B)$ for modules A, B over a ring R, is to consider elements C fitting into a short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0,$$

up to equivalence $C \sim D$ if there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "\wr"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & D \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

It is possible to define a sum $C + D$ in $\mathrm{Ext}^1_R(A, B)$ where the identity element E corresponds to the split extension where $E = A \oplus B$. Higher Ext-groups can be interpreted similarly, but with n terms fitting between A and B in the exact sequence. For now, we will only view Ext^2 as what follows after Ext^1 in the long exact sequence of Ext-groups coming from the right resolution of the Hom-functor.

Proof of Theorem 3.2. The base extension $\mathrm{Spec}(k) \subset \mathrm{Spec}(D)$ defines a short exact sequence

$$0 \longrightarrow k \xrightarrow{\times\epsilon} D \longrightarrow k \longrightarrow 0, \tag{5}$$

and hence a sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X\times D} \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{6}$$

by flatness of $\mathcal{O}_X$ over k.

Tensoring this sequence with a given deformation $\mathcal{F}$ gives the exact sequence, since $\mathcal{F}$ is flat, of $\mathcal{O}_{X\times D}$-modules

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X\times D}} \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X\times D}} \mathcal{O}_X \longrightarrow 0.$$

We recognize the outer terms as being $\mathcal{E}$ by the definition of a deformation. Thus, the sheaf $\mathcal{F}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{E} \longrightarrow 0. \tag{7}$$

Noting that these sheaves are also O_X-modules, we obtain, as in Remark 3.3, an element of $\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})$.

Conversely, an element of $\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})$ gives an extension $\mathcal{F}$ as in (7). We just need to make $\mathcal{F}$ into an $\mathcal{O}_{X\times D}$-module by defining multiplication by ϵ. Motivated by equation (5), we simply project $\mathcal{F}$ onto $\mathcal{E}$ by the map g, followed by the injection of $\mathcal{E}$ into $\mathcal{F}$ by f. In symbols, with a slight abuse of notation, we have the action of $\mathcal{O}_{X\times D} = \mathcal{O}_X \oplus (\epsilon)$ on $\mathcal{F}$ by $(s, c\epsilon)(x) = sx + cf(g(x))$ for $s, c \in \mathcal{O}_X$ and $x \in \mathcal{F}$ (as sections over an open set). The multiplication on $\mathcal{O}_X \oplus (\epsilon)$ is here given by $(x_1, c_1\epsilon) \cdot (x_2, c_2\epsilon) = (x_1x_2, (x_1c_2 + x_2c_1)\epsilon)$.

Since the obtained sheaf $\mathcal{F}$ can be shown to be flat over D ([**Th00a**, Lemma 3.7]), this indeed gives the desired bijection. $\square$

Continuing with obstructions, we now consider base extensions

$$\mathrm{Spec}(k) \subset B = \mathrm{Spec}(R) \subset B' = \mathrm{Spec}(R')$$

for local Artinian k-algebras R, R'. We let the ideal of $\mathrm{Spec}(k)$ in B be $\mathfrak{m}$ and the ideals of B in B' resp. $\mathrm{Spec}(k)$ in B' be I and $\mathfrak{n}$, respectively. Furthermore, recalling that we are considering small lifts of deformations one step at a time, we require that $I \cdot \mathfrak{m} = 0$, which in particular means that $I^2 = 0$. This condition also ensures that I becomes a vector space over $k = R/\mathfrak{m}$.

We obtain the short exact sequence

$$0 \longrightarrow I \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{n} \longrightarrow 0.$$

For a given deformation $\mathcal{E}$ of a coherent sheaf $\mathcal{E}_0$ on X to $X \times B$, we would like to see whether the deformation can be extended to a sheaf $\mathcal{F}$ over $X \times B'$.

Considering the Ext-sequence obtained by applying $\mathrm{Hom}(\mathcal{E}_0, \_)$ to the short exact sequence

$$0 \longrightarrow \mathcal{E}_0 \otimes I \longrightarrow \mathcal{E} \otimes \mathfrak{m} \longrightarrow \mathcal{E} \otimes \mathfrak{n} \longrightarrow 0,$$

we obtain the following result (for details, see [**Th00a**, p.12-13]):

Theorem 3.4. *With the above notation, let $\mathcal{E}$ be a deformation of a coherent sheaf $\mathcal{E}_0$ on X to $X \times B$. The sheaf $\mathcal{E}$ defines a class $\delta_{\mathcal{E}} \in \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \otimes I$, which vanishes if and only if $\mathcal{E}$ can be extended to a sheaf $\mathcal{F}$ over $X \times B'$.*

If we can identify the sheaves which we want to deform, we now have good candidates for the local tangent- and obstruction spaces $T_p\mathcal{M}$ and Ob_p introduced in Chapter 2, namely the groups Ext^1 and Ext^2. Globally, a bit more work is required, see [**Th00a**, pp. 20-12]. For Hilbert schemes, it will be enough to consider coherent sheaves of rank 1 with trivial determinant. We will explain this in the coming chapter, before introducing and giving properties of the DT invariants.

For stable pairs, it is easier to carry out the deformation theory in the derived category of coherent sheaves, which will be introduced later. But the idea remains the same, and the Ext-groups will still be central.

CHAPTER 4

Donaldson-Thomas invariants

Fruits of the ideal world on a
Calabi-Yau threefold.

Recall that we are counting curves in a class (β, n), for $\beta \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_X) = n$, on a CY 3-fold X. The *Hilbert scheme*

$$I_n(X,\beta) = \left\{ \begin{array}{l} \text{closed subschemes } Z \subset X \text{ of } \dim(Z) \leq 1, \text{with} \\ \chi(\mathcal{O}_Z) = n \text{ and class } \beta \in H_2(X,\mathbb{Z}) \end{array} \right\}$$

is the compact moduli space from which DT invariants will be constructed. For constructing the virtual fundamental class, it turns out that an alternative interpretation of the Hilbert scheme works better. A closed subscheme $Z \subset X$ is associated with an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, which is coherent since X is Noetherian. To make a practical classification of the ideal sheaves corresponding to closed subschemes $Z \subset X$ of $\dim(Z) \leq 1$, we introduce some properties and quantities associated to coherent sheaves.

4.1. Vocabulary of coherent sheaves

Definition 4.1. Let $\mathcal{F}$ be a coherent sheaf on a projective variety X over k. The *Euler characteristic* of $\mathcal{F}$ is the alternating sum $\chi(\mathcal{F}) = \sum_{i=1}^{n}(-1)^i \dim_k H^i(X, \mathcal{F})$, where $n = \dim(X)$.

A fixed embedding $X \hookrightarrow \mathbb{P}^N$ corresponds to a very ample line bundle $\mathcal{O}_X(1)$ on X, which determines the hyperplane sections of X inside $\mathbb{P}^N$. For a coherent sheaf $\mathcal{F}$, the assignment $m \mapsto \chi(\mathcal{F} \otimes \mathcal{O}_X(m))$ for $m \in \mathbb{Z}$ determines a polynomial $P(\mathcal{F}, m)$ in m for sufficiently large values of m. The resulting polynomial $P(\mathcal{F}, t)$ is the *Hilbert polynomial* of $\mathcal{F}$ with respect to the given embedding. The degree of $P(\mathcal{F}, t)$ equals the dimension $\dim(\mathcal{F}) := \dim(\mathrm{Supp}(\mathcal{F}))$.

Definition 4.2. Again, let $\mathcal{F} \neq 0$ be a coherent sheaf on a projective variety X over k, where $\dim(X) = d$. Denote by $a_d(\mathcal{F})$ the d^{th} coefficient of $P(\mathcal{F}, t)$, possibly equal to zero. The *rank* of $\mathcal{F}$ is defined to be $\mathrm{rk}(\mathcal{F}) = a_d(\mathcal{F})/a_d(\mathcal{O}_X)$. Since $\dim(\mathcal{F}) \leq \dim(X)$, we have $\mathrm{rk}(\mathcal{F}) = 0$ if and only if $\dim(\mathcal{F}) < d$.

Definition 4.3. Assume further that X is smooth, so that $\mathcal{F}$ has a finite resolution of locally free sheaves

$$0 \longrightarrow \mathrm{E}_l \longrightarrow \ldots \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0, \tag{8}$$

where $l = \dim(X)$ (this is Hilbert's syzygy theorem).

The *determinant* of $\mathcal{F}$ is defined to be the line bundle $\bigotimes_{i=0}^{l} \Lambda^{\mathrm{rk}(\mathrm{E}_i)} \mathrm{E}_i^{(-1)^i}$.

Example 4.4. As a sanity check, we compute the rank of the free vector bundle $\mathcal{F} = \mathcal{O}_X^{\oplus r}$. Since χ is additive over short exact sequences, so is the Hilbert polynomial and hence the rank. So $\mathrm{rk}(\mathcal{F}) = r \cdot \mathrm{rk}(\mathcal{O}_X) = r$.

More generally, the rank of a locally free sheaf $\mathcal{E}$ can be computed on an open set $U \subset X$ where $\mathcal{E}|_U$ is free, since X is connected. Resolving a coherent sheaf by a finite number of locally free sheaves as in (8), we can calculate the rank of $\mathcal{F}$ by additivity over the sequence. This means that the rank of $\mathcal{F}$ is independent of the projective embedding of X.

4.2. A characterization of $I_n(X,\beta)$

In the following example, we find some properties of the ideal sheaves in our Hilbert scheme that will help us characterize them better.

Example 4.5. Let X be a smooth projective 3-fold over k, and $\mathcal{I}_Z \subset \mathcal{O}_X$ an ideal sheaf corresponding to a subscheme Z of $\dim(Z) \leq 1$. Since X is integral, the rings $\mathcal{O}_X(U)$ are integral domains for open sets $U \subset X$ by definition. Thus, since $\mathcal{I}_Z$ is a subsheaf of the free sheaf $\mathcal{O}_X$, the sheaf $\mathcal{I}_Z$ is torsion free. Moreover, consider the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0, \tag{9}$$

where $\dim(\mathcal{O}_Z) \leq 1 < \dim(X)$. Additivity of rank over this sequence gives $\mathrm{rk}(\mathcal{I}_Z) = \mathrm{rk}(\mathcal{O}_X) = 1$.

We next find the determinant of $\mathcal{I}_Z$. The sequence (9) gives us the relation $\det(\mathcal{O}_X) = \det(\mathcal{I}_Z) \otimes_{\mathcal{O}_X} \det(\mathcal{O}_Z)$. If we can prove that $\det(\mathcal{O}_Z) \cong \mathcal{O}_X$, it will also follow that $\det(\mathcal{I}_Z) \cong \mathcal{O}_X$. Using the resolution (8) for $\mathcal{F} = \mathcal{O}_Z$, since $\mathrm{codim}(\mathcal{O}_Z) \geq 2$, the sequence

$$0 \longrightarrow \mathrm{E}_l \longrightarrow \ldots \longrightarrow \mathrm{E}_1 \longrightarrow \mathrm{E}_0 \longrightarrow 0 \tag{10}$$

is exact away from a locus of codimension ≥ 2. Thus, the determinant

$$\det(\mathcal{O}_Z) = \bigotimes_{i=0}^{l} \Lambda^{\mathrm{rk}(\mathrm{E}_i)} \mathrm{E}_i^{(-1)^i}$$

is trivial away from a (closed) set of codimension ≥ 2.

From the isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(U)$ when $U \subset X$ is open and $X \backslash U$ has codimension ≥ 2 (X is a smooth variety, so $\mathrm{Pic}(X) = \mathrm{Cl}(X)$), we deduce that $\det(\mathcal{O}_Z)$ and hence $\det(\mathcal{I}_Z)$ is trivial.

We now obtain a new way of recognizing ideal sheaves:

Lemma 4.6. *Let X be a projective variety over k. Then we have the alternative description* [1]

$$I_n(X,\beta) = \left\{ \begin{array}{l} \text{coherent sheaves } \mathcal{F} \text{ with } \mathrm{rk}(\mathcal{F}) = 1,\ \det(\mathcal{F}) \cong \mathcal{O}_X, \\ \chi(\mathcal{F}) = -n \text{ and } \mathrm{ch}_2(\mathcal{F}) = -\beta \in H_2(X,\mathbb{Z}) \end{array} \right\}.$$

---

[1] See [**EH16**, p.485] for the definitions of the Chern characters ch_i, and [**Br11**, p.5] for seeing why we can view them as taking values in homology with integral coefficients rather than in cohomology with rational coefficients.

We have already seen that the ideal sheaves corresponding to closed subschemes of $\dim(Z) \leq 1$ satisfy this description. See [**Th00a**, p.26] for the converse direction.

We now turn to describing the construction of the virtual fundamental class $[I_n(X,\beta)]^{vir}$ associated to $I_n(X,\beta)$. From the virtual fundamental class, we will then obtain the DT invariants.

4.3. Deformation theory of $I_n(X,\beta)$

In a way analogous to the deformation of coherent sheaves, we can confine ourselves to deforming coherent sheaves of a fixed rank and determinant. For us, this means choosing the rank to be 1 and the determinant to be $\mathcal{O}_X$. By our discussion above, this will correspond to deforming ideal sheaves associated to closed subschemes of dimension ≤ 1.

Informally, first order changes in the determinant are detected by the trace map, which for locally free sheaves E is a map defined by $\operatorname{tr}\colon \mathscr{H}om\,(E,E) \longrightarrow \mathcal{O}_X$. Composing with the identity map in the opposite direction, we obtain a splitting $\mathscr{H}om\,(E,E) = \mathscr{H}om\,(E,E)_0 \oplus \mathcal{O}_X$, where $\mathscr{H}om\,(E,E)_0$ is the kernel of tr. In cohomology, since Ext is the right derived functor of $\mathscr{H}om$, we obtain splittings

$$\operatorname{Ext}^i(E,E) = \operatorname{Ext}^i_0(E,E) \oplus \mathrm{H}^i(X,\mathcal{O}_X)$$

for $i \geq 0$.

For a coherent sheaf $\mathcal{E}$, the map will be defined in terms of chain complexes arising from the resolution of locally free sheaves (for details, see [**Th00a**, p.14]). We similarly obtain splittings

$$\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = \operatorname{Ext}^i_0(\mathcal{E},\mathcal{E}) \oplus \mathrm{H}^i(X,\mathcal{O}_X). \tag{11}$$

The tangent and obstruction spaces for the ideal sheaves $\mathcal{I}$ in $I_n(X,\beta)$ can be shown to be $\operatorname{Ext}^1_0(\mathcal{I},\mathcal{I})$ resp. $\operatorname{Ext}^2_0(\mathcal{I},\mathcal{I})$. Using the notation in the sequence (4), we obtain $T_p\mathcal{M} = \operatorname{Ext}^1_0(\mathcal{I},\mathcal{I})$ and $\mathrm{Ob}_p = \operatorname{Ext}^2_0(\mathcal{I},\mathcal{I})$, where $\mathcal{M}$ is the moduli space $I_n(X,\beta)$ and p is the point in $\mathcal{M}$ corresponding to the ideal sheaf $\mathcal{I}$.

For the construction of the locally free sheaves E_1 and E_2 in (4), see [**HT10**]. Equipped with all terms in the sequence (4), we obtain a virtual fundamental class $[I_n(X,\beta)]^{vir}$ by generalizing the methods from the vector bundle situation described in Chapter 2 (using techniques in [**BF97**] or [**LT98**]). The virtual fundamental class will be invariant upon deforming X, in the sense of [**Th00a**, Cor. 3.53].

We now compute the virtual dimension of $I_n(X,\beta)$.

Claim 4.7. The virtual dimension of $I_n(X,\beta)$ equals 0.

Proof. Here it is essential that X is a Calabi-Yau 3-fold. Serre duality applied to coherent sheaves on a smooth projective 3-fold gives an isomorphism

$$\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = \operatorname{Ext}^{3-i}(\mathcal{E},\mathcal{E}\otimes K_X)^\vee$$

for a coherent sheaf $\mathcal{E}$ on X and $0 \leq i \leq 3$.

Since $K_X \cong \mathcal{O}_X$ for a CY 3-fold, we deduce that $\operatorname{Ext}^1(\mathcal{I},\mathcal{I}) = \operatorname{Ext}^2(\mathcal{I},\mathcal{I})^\vee$ for ideal sheaves $\mathcal{I} \in I_n(X,\beta)$.

Again by Serre duality and the Calabi-Yau condition, we have

$$H^1(X,\mathcal{O}_X) \cong \operatorname{Ext}^2(\mathcal{O}_X,\mathcal{O}_X)^\vee \cong \operatorname{Ext}^1(\mathcal{O}_X,\mathcal{O}_X) \cong H^2(X,\mathcal{O}_X)^\vee.$$

Thus, from the splittings (11), we obtain

$$\operatorname{vdim}(I_n(X,\beta)) = \dim(\operatorname{Ext}^1_0(\mathcal{I},\mathcal{I})) - \dim(\operatorname{Ext}^2_0(\mathcal{I},\mathcal{I})) = 0. \qquad \square$$

Since the virtual fundamental class has dimension 0, we are able to "count" its components, i.e. taking the *degree* of the 0-cycle $[I_n(X,\beta)]^{vir} \in \mathcal{A}_0(I_n(X,\beta))$ (see [**Fu98**, p.13] for a complete definition of degree, but think of it as counting the irreducible subschemes with correct multiplicities).

The resulting integer will act as a virtual count for the objects in $I_n(X,\beta)$, and hence for the curves in the class (β, n). This is the main advantage, worth restating in a memorable way: *A Calabi-Yau 3-fold is expected to have a finite number of curves in a given class, allowing us to create a virtual count.*

Explicitly, we define the DT invariants as follows:

Definition 4.8. For a given curve class (β, n) on a Calabi-Yau 3-fold X, the corresponding *Donaldson-Thomas invariant* is defined as $I_{\beta,n} = \deg([I_n(X,\beta)]^{vir})$.

If $I_n(X,\beta)$ is smooth, the virtual class will agree with the fundamental class $[I_n(X,\beta)]$. In this case, we have $I_{\beta,n} = \deg([I_n(X,\beta)]) = \chi(I_n(X,\beta))$, a consequence of the Hirzebruch-Riemann-Roch Theorem. In the paper [**Be09**], Behrend introduces a way of generalizing this result to any CY 3-fold X. For a proper moduli space $\mathcal{M}$ with a *symmetric obstruction theory* (see [**BF08**]), Behrend defines a so-called *constructible function* $\nu\colon \mathcal{M} \to \mathbb{Z}$. This function takes the constant value $(-1)^{\dim(\mathcal{M})}$ at smooth points of $\mathcal{M}$, and satisfies further properties as in [**Be09**, p.4]. Defining the *weighted Euler characteristic* of $\mathcal{M}$ as $\chi(\mathcal{M},\nu) = \sum_{n\in\mathbb{Z}} n\chi(\nu^{-1}(n))$, being equal to $(-1)^{\dim(\mathcal{M})}\chi(\mathcal{M})$ when $\mathcal{M}$ is smooth, Behrend observed the following result:

Theorem 4.9. *The DT invariants satisfy* $I_{\beta,n} = \chi(I_n(X,\beta),\nu)$.

Using this interpretation, we obtain an explicit description of the DT invariants that allows us to manipulate them algebraically. For instance, this lets us establish a certain relation between the DT- and PT invariants, which is the content of Chapter 7.

It is also possible to construct the virtual fundamental class in the language of derived categories, which will be the topic of the next chapter. While we do not revisit the construction of the DT invariants in this setting, we will work entirely in the derived language to construct the PT invariants. The methods will nevertheless be very similar.

CHAPTER 5

The derived category of coherent sheaves

> *With a true appreciation of this topic, you might be deriving Hom for Christmas...*

While arbitrary (quasi)coherent sheaves can be difficult to work with, certain classes of them are often better adapted to specific situations. For example, when using the tensor product, we prefer to work with locally free coherent sheaves since they are flat.

We can often replace a coherent sheaf $\mathcal{F}$ with a collection of better behaved sheaves $(\mathcal{G}_0)_{i\geq 0}$ by considering either a *left resolution*, i.e. an exact sequence of the form

$$\ldots \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

or a *right resolution* by sheaves $(\mathcal{G}^0)_{i\geq 0}$, which is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \mathcal{G}^2 \longrightarrow \ldots .$$

A coherent sheaf admits several resolutions, each of which might be well suited for a particular context. For instance, when computing sheaf cohomology, resolutions of interest might be the Čech resolution or a flasque resolution. Moreover, any coherent sheaf on a projective variety admits a finite length left resolution by locally free coherent sheaves.

In the derived category, we will be able to freely replace a (quasi)coherent sheaf with resolutions of better behaved sheaves. Starting with the category of complexes of (quasi)coherent sheaves up to homotopy equivalence, the derived category is obtained by imposing further relations that will imply that a sheaf is "isomorphic" to its resolution.

More precisely, we consider the *homotopy category of complexes* of quasicoherent sheaves on a projective variety X, denoted $K(\mathbf{QCoh}(X))$, whose objects are complexes $(X^\bullet, d)$ of quasicoherent sheaves

$$(X^\bullet, d) = \ \ldots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \ldots$$

where $d^n \circ d^{n+1} = 0$ for all $n \in \mathbb{Z}$.

The morphisms of $K(\mathbf{QCoh}(X))$ consist of chain maps, where we further declare homotopy equivalent maps to be equal as morphisms in $K(\mathbf{QCoh}(X))$.

To obtain the *derived category* $\mathcal{D}(\mathbf{QCoh}(X))$, we impose even more equivalences on the morphisms in $K(\mathbf{QCoh}(X))$. We consider so-called *quasi-isomorphisms*, by which we refer to chain maps $X^\bullet \longrightarrow Y^\bullet$ which induce isomorphisms in (chain) cohomology. The derived category $\mathcal{D}(\mathbf{QCoh}(X))$ is obtained from $K(\mathbf{QCoh}(X))$ by formally assigning an inverse to each quasi-isomorphism in $K(\mathbf{QCoh}(X))$, turning the quasi-isomorphisms into isomorphisms.

Having imposed these equivalence relations on morphisms of complexes, we lose some favorable properties and structures that normally accompany chain maps. For instance, it is no longer clear what a kernel or a cokernel of a morphism is, or whether we can obtain long exact sequences in cohomology.

5.1. The triangulated category

It turns out that we can construct analogous structures in $\mathcal{D}(\mathbf{QCoh}(X))$, by considering $\mathcal{D}(\mathbf{QCoh}(X))$ as a *triangulated category.* For this, we introduce a functor T, called the *shift functor*, which shifts the degrees of a complex to the right, or if you prefer, shifts each term in the complex a step to the left. However, the chain maps will change sign. Explicitly, for a complex $X^\bullet$, we have $T(X)^n = X^{n+1}$, and $d_{T(X)} = -d_X$. We usually denote the complex obtained by applying the shift functor n times to $X^\bullet$ by $X^\bullet[n]$. Similarly, we can shift the degrees n steps to the left and denote the resulting complex by $X^\bullet[-n]$.

Moreover, we introduce the notion of *distinguished triangles*, which are defined as follows:

Consider a chain map $f\colon X^\bullet \longrightarrow Y^\bullet$, to which we associate a complex $C^\bullet(f)$ called the *mapping cone* of f, defined by the terms $C^i(f) = X^{i+1} \oplus Y^i$ and the boundary maps $d^i_{C(f)} = \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}$.

We have natural maps $g\colon Y^\bullet \longrightarrow C^\bullet(f)$, and $h\colon C^\bullet(f) \longrightarrow X^\bullet[1]$, , given by the inclusion of Y^i into $X^{i+1} \oplus Y^i$ and the projection of $X^{i+1} \oplus Y^i$ onto X^{i+1}, respectively.

The topologically minded reader might want to compare this construction to the mapping cone of a map $f\colon X \longrightarrow Y$ between topological spaces, which is a space C_f formed by attaching the mapping cylinder $(X \times I) \sqcup_f Y$ of f to Y by identifying the points $(x,1) \sim f(x)$ for $x \in X$ and contacting the set $\{(x,0) |\ x \in X\}$ to a point.

The tuple $(X^\bullet, Y^\bullet, C^\bullet(f), f, g, h)$ will be called a *distinguished triangle*, which deserves its name from the alternative notation

$$\begin{tikzcd}
 & C^\bullet(f) \arrow[dl, dashed] & \\
X^\bullet \arrow[rr] & & Y^\bullet, \arrow[ul]
\end{tikzcd}$$

where the dashed arrow indicates a morphism from $C^\bullet(f)$ to $X^\bullet[1]$.

Moreover, suppose that we have a commutative diagram

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d] & Y^\bullet \arrow[r, "g"] \arrow[d] & C^\bullet(f) \arrow[r, "h"] \arrow[d] & A^\bullet[1] \arrow[d] \\
A^\bullet \arrow[r, "u"] & B^\bullet \arrow[r, "v"] & C^\bullet \arrow[r, "w"] & A^\bullet[1]
\end{tikzcd}$$

in $\mathcal{D}(\mathbf{QCoh}(X))$, where the tuple $(X^\bullet, Y^\bullet, C^\bullet(f), f, g, h)$ forms an exact triangle and the vertical maps are isomorphisms. Then the tuple $(A^\bullet, B^\bullet, C^\bullet, u, v, w)$ will also be classified as a distinguished triangle.

The complex $C^\bullet[-1]$ then serves as a kernel for the morphism u, while $C^\bullet$ serves as a cokernel. Furthermore, as we wished for, any distinguished triangle $(A^\bullet, B^\bullet, C^\bullet, u, v, w)$, yields a long exact sequence in cohomology

$$\begin{tikzcd}
\dots \arrow[r] & H^{n-1}(C^\bullet) \arrow[r] & H^n(A^\bullet) \arrow[r] & H^n(B^\bullet) \arrow[r] & H^n(C^\bullet) \arrow[dllll, rounded corners] \\
H^{n+1}(A^\bullet) \arrow[r] & H^{n+1}(B^\bullet) \arrow[r] & H^{n+1}(C^\bullet) \arrow[r] & H^{n+2}(A^\bullet) \arrow[r] & \dots .
\end{tikzcd} \tag{12}$$

We will mostly be concerned with the *bounded derived category* of coherent sheaves on a variety X, denoted $\mathcal{D}^b(X)$, which is the subcategory of $\mathcal{D}(\mathbf{QCoh}(X))$ consisting of bounded length complexes of coherent sheaves.

A quasicoherent sheaf $\mathcal{F}$ can be considered as an object of $\mathcal{D}(\mathbf{QCoh}(X))$ by viewing it as a complex concentrated in degree 0. In Example 5.1 below, we justify our promise that quasicoherent sheaves should be interchangeable with their resolutions in $\mathcal{D}(\mathbf{QCoh}(X))$.

Example 5.1. Consider a left resolution

$$\dots \longrightarrow \mathcal{G}_2 \xrightarrow{g_2} \mathcal{G}_1 \xrightarrow{g_1} \mathcal{G}_0 \xrightarrow{g_0} \mathcal{F} \longrightarrow 0$$

of a quasicoherent sheaf $\mathcal{F}$. Then we have a chain map

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d] & \mathcal{G}_2 \arrow[r, "g_2"] \arrow[d] & \mathcal{G}_1 \arrow[r, "g_1"] \arrow[d] & \mathcal{G}_0 \arrow[r] \arrow[d, "g_0"] & 0 \arrow[d, equal] \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

inducing an isomorphism in cohomology. This means that $\mathcal{F}$ is isomorphic to the complex $\dots \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_0 \longrightarrow 0$ in $\mathcal{D}(\mathbf{QCoh}(X))$. A similar argument works if we instead consider a right resolution.

In particular, for a short exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \longrightarrow 0$$

of quasicoherent sheaves, the cokernel $\mathcal{G}$, viewed as an object of $\mathcal{D}^b(X)$, is isomorphic to the complex $[\mathcal{E} \xrightarrow{f} \mathcal{F}]$ formed by the first two terms.

5.2. Derived functors

There is yet an important issue to overcome before we can start working with the derived category. Let us see what happens if we naively try to apply our favorite endofunctors from the category $\mathbf{QCoh}(X)$ of quasicoherent sheaves. Consider the functor $F = \_ \otimes \mathcal{G}$ for a quasicoherent $\mathcal{G}$. Our natural instinct would be to define $F(A^\bullet) = (A^i \otimes \mathcal{G})_{i\in\mathbb{Z}}$ as a functor in $\mathcal{D}(\mathbf{QCoh}(X))$. However, suppose that a quasicoherent sheaf $\mathcal{F}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow 0.$$

Then $\mathcal{F}$ is isomorphic to the complex $[\mathcal{F}_2 \longrightarrow \mathcal{F}_1]$ in $\mathcal{D}(\mathbf{QCoh}(X))$. But, comparing cohomologies, we see that the complex $[\mathcal{F}_2 \otimes \mathcal{G} \longrightarrow \mathcal{F}_1 \otimes \mathcal{G}]$ is not in general isomorphic to $\mathcal{F} \otimes \mathcal{G}$ in $\mathcal{D}(\mathbf{QCoh}(X))$, unless $\mathcal{G}$ is flat!

Instead, we try find to find suitable alternatives for $\mathcal{D}(\mathbf{QCoh}(X))$ to the functors we are interested in. Since we would prefer to work in $\mathcal{D}^b(X)$, we further hope that the functors we construct on $\mathcal{D}(\mathbf{QCoh}(X))$ will restrict to functors on $\mathcal{D}^b(X)$.

The functors we will be concerned with here are the following: Hom, $\otimes$ and the inverse image f^* for $f\colon X \to Y$ a morphism of projective varieties.

First, we deal with Hom which is a left exact functor. For this, we make use of the fact that any quasicoherent sheaf on a variety X admits a right resolution by injective objects in $\mathbf{QCoh}(X)$, i.e. quasicoherent sheaves $(I^i)_{i\geq 0}$ with the functors $\mathrm{Hom}(\_, I^i)$ being exact (see [Ha66, II.7.18]). In turn, this implies that any object in $\mathcal{D}^+(\mathbf{QCoh}(X))$ (here $\mathcal{D}^+$ refers to complexes bounded below) is isomorphic to a complex consisting of injectives ([Hu06, Prop. 2.35]).

Consider the functor $\mathrm{Hom}^\bullet(A^\bullet, \_)\colon \mathcal{D}^+(\mathbf{QCoh}(X)) \to \mathcal{D}(\mathbf{Ab})$ (here $\mathbf{Ab}$ denotes the category of abelian groups) given by

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) = \prod_{k\in\mathbb{Z}} \mathrm{Hom}(A^k, B^{k+i}),\ d^i = \prod_{k\in\mathbb{Z}} (d_A^k + (-1)^{i+1} d_B^{k+i}).$$

For $B^\bullet$ in $\mathcal{D}(\mathbf{QCoh}(X))$, we can define the *right derived functor*

$$\mathbf{R}\mathrm{Hom}^\bullet(\_, B^\bullet)\colon \mathcal{D}^+(\mathbf{QCoh}(X)) \to \mathcal{D}(\mathbf{Ab})$$

of the functor $\mathrm{Hom}^\bullet(\_, B^\bullet)$ by defining $\mathbf{R}\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) := \mathrm{Hom}^\bullet(I^\bullet, B^\bullet)$, where $I^\bullet$ is a complex of injectives isonorphic to $A^\bullet$ in $\mathcal{D}^+(\mathbf{QCoh}(X))$. The definition will be independent of the choice of $I^\bullet$, see [**Ha66**, p.53]. Restricting to $\mathcal{D}^b(X)$, we obtain a functor $\mathbf{R}\mathrm{Hom}^\bullet(\_, \_)\colon \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}(\mathbf{Ab})$.

We can now define Ext groups on $\mathcal{D}^b(X)$ by taking cohomology:

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) := H^i(\mathbf{R}\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)).$$

These Ext groups can be shown to satisfy $\mathrm{Ext}^i(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{D}^b(X)}(A^\bullet, B^\bullet[i])$. Moreover, for (quasi)coherent sheaves, the Ext-groups in the derived category agree with the usual Ext-groups for quasicoherent sheaves ([Ha66, Thm I.6.4]).

It would seem natural that projective resolutions, i.e. left resolutions of coherent sheaves P with $\mathrm{Hom}(P, \_)$ exact, would serve as counterparts for defining left derived functors from right exact functors on $\mathbf{QCoh}(X)$. However, there are nearly never enough projective objects in $\mathbf{QCoh}(X)$ to be able to find projective resolutions for all quasicoherent sheaves on X.

Instead, for deriving the functors $\otimes$ and f^*, it will be enough to consider resolutions of locally free sheaves ([Hu06, pp.78,81]). Since a coherent sheaf has a finite resolution of locally free sheaves, we can form a derived tensor product on $\mathcal{D}^b(X)$ by applying the usual tensor product to a resolution of locally free sheaves.

Precisely, we start with the functor

$$\_ \otimes B^\bullet\colon K^b(X) \to K^b(X)$$

given by

$$(A^\bullet \otimes B^\bullet)^i = \bigoplus_{k\in\mathbb{Z}} A^k \otimes B^{i-k},\ d^i = d_A \otimes 1 + (-1)^i 1 \otimes d_B.$$

We obtain the *left derived tensor product*

$$\_ \otimes^{\mathbf{L}} \_\colon \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

by defining $A^\bullet \otimes^{\mathbf{L}} B^\bullet := E^\bullet \otimes^{\mathbf{L}} B^\bullet$ where $E^\bullet$ is a finite length complex of locally free sheaves isomorphic to $A^\bullet$ in $\mathcal{D}^b(X)$.

Similarly, for a morphism $f\colon X \longrightarrow Y$ of projective varieties, it is possible to define the *left derived inverse image* functor

$$\mathbf{L}f^* : f^{-1}(\_) \otimes^{\mathbf{L}}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X),$$

with the subtlety that we are now deriving the tensor product for $f^{-1}\mathcal{O}_Y$-modules rather than $\mathcal{O}_X$-modules.

A right derived functor $\mathbf{R}F$ will take a distinguished triangles $(A^\bullet, B^\bullet, C^\bullet, u, v, w)$ in $\mathcal{D}^b(X)$ to another distinguished triangle, thus giving rise to a long exact sequence in cohomology, denoting $H^i(\mathbf{R}F(A^\bullet))$ by $R^iF(A^\bullet)$:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{n-1}F(C^\bullet) \arrow[r] & RF^n(A^\bullet) \arrow[r] & R^nF(B^\bullet) \arrow[r] & R^nF(C^\bullet) \arrow[dllll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
R^{n+1}F(A^\bullet) \arrow[r] & R^{n+1}F(B^\bullet) \arrow[r] & R^{n+1}F(C^\bullet) \arrow[r] & R^{n+2}F(A^\bullet) \arrow[r] & \cdots .
\end{tikzcd}$$

We obtain a similar sequence, but reversed, for the homologies $L^iF(A^\bullet) := H_i(\mathbf{L}F(A^\bullet))$ of a left derived functor $\mathbf{L}F$.

In the coming chapter, we will use the tools just developed to deform objects in the moduli space of stable pairs.

CHAPTER 6

Pandharipande-Thomas invariants

Insert your own quote here!

We now consider the moduli space of *stable pairs* as an alternative compactification of $\mathcal{M}(\beta, n)$ associated to a Calabi-Yau 3-fold X:

$$P_n(X,\beta) = \left\{ \begin{array}{l} \text{Pairs } (\mathcal{F}, s) \text{ of } \mathcal{F} \text{ a pure 1-dimensional coherent sheaf on } X \\ \text{with } [\mathrm{Supp}(\mathcal{F})] = \beta,\ \chi(\mathcal{F}) = n \text{ and} \\ s \text{ a section satisfying } \dim(\mathrm{coker}(s)) = 0. \end{array} \right\}$$

We begin with exploring the geometry of stable pairs.

6.1. The geometry of stable pairs

We make use of the notion *scheme theoretic support* $\mathrm{Supp}(\mathcal{E})$ of a sheaf $\mathcal{E}$ on X, defined to be the closed subscheme whose structure sheaf, when restricted to affine open sets $U = \mathrm{Spec}(A) \subset X$, is the sheaf associated to the module $A/\mathrm{Ann}(\mathcal{E}|_U)$.

Claim 6.1. The scheme theoretic support C of $\mathcal{F}$ has no embedded points.

Proof. We break up the proof into three parts:

1. $\mathrm{Supp}(\mathcal{F}) = \mathrm{Supp}(\mathrm{im}(s)) = C$;
2. $\mathrm{im}(s) \cong \mathcal{O}_C$ where $\mathcal{O}_C$ is the structure sheaf of C;
3. im(s) has no embedded points.

To prove 1, we restrict ourselves to an affine open set $U = \mathrm{Spec}(A) \subset X$ where $\mathcal{F}|_U = \tilde{M}$ is the sheaf of a module and $s \in M$. We will prove that $\mathrm{Ann}(s) = \mathrm{Ann}(M)$. The inclusion $\mathrm{Ann}(M) \subset \mathrm{Ann(s)}$ is immediate. For the reverse inclusion, suppose there are elements $a \in A$, $f \in M$ with $as = 0$ but $af \neq 0$. Then, for each point $\mathfrak{p} \notin \mathrm{Supp}(\mathrm{coker}(s))$, the module $M_\mathfrak{p}$ is generated by $s_\mathfrak{p}$. Thus, we have $f_\mathfrak{p} = bs_\mathfrak{p}$ for some $b \in A_\mathfrak{p}$. Multiplying both sides by $a_\mathfrak{p}$, we see that af is not supported at $\mathfrak{p}$. Hence, the element $af \in M$ generates a nonzero submodule $N \subset M$ whose support lies in the 0-dimensional space $\mathrm{Supp}(\mathrm{coker}(s))$. Extending the sheaf $\tilde{N}$ by zero outside U, we obtain a nonzero subsheaf of $\mathcal{F}$ supported in dimension 0, a contradiction to $\mathcal{F}$ being pure.

For 2, since the map $\mathcal{O}_X \xrightarrow{s} \mathrm{im}(s)$ is surjective, the sheaf $\mathrm{im}(s)$ is the structure sheaf of its scheme theoretic support, which is C by 1.

For 3, we note that $\mathrm{im}(s)$ is pure as a subsheaf of $\mathcal{F}$. We restrict to an affine open $U = \mathrm{Spec}(A)$ of X. Let S denote the submodule of $\mathcal{F}|_U$ spanned by s. Assume that S contains primes $\mathfrak{p} = \mathrm{Ann}(m), \mathfrak{q} = \mathrm{Ann}(n)$ for nonzero $m, n \in S$ with $\mathfrak{p} \subset \mathfrak{q}$. As topological spaces, we then have $\mathrm{Supp}(n) = V(\mathfrak{q}) \subset V(\mathfrak{p}) = \mathrm{Supp}(m)$. If $\mathfrak{p} \neq \mathfrak{q}$, then $\dim(V(\mathfrak{q})) < \dim(V(\mathfrak{p}))$. This implies that the submodule of S spanned by n is supported in dimension 0, a contradiction to $\operatorname{im}(s)$ being pure.

Combining $1-3$ proves the claim. $\square$

As a conclusion, we can associate to each stable pair (F, s) a pure 1-dimensional closed subscheme $C = \operatorname{Supp}(F)$ with no embedded points, and a finite set of points $\operatorname{Supp}(\operatorname{coker}(s))$ lying on C. Although there can be several stable pairs with the same supports of F resp. $\operatorname{coker}(s)$ ([**PT09**, p.9]), we have still reduced the size of the moduli space significantly compared to the Hilbert scheme. A reason for this is that the Hilbert scheme allows for 0-dimensional points which are not embedded in any 1-dimensional component.

As a result of Claim (6.1), we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow Q \longrightarrow 0, \tag{13}$$

where $Q = \operatorname{coker}(s)$ and $\mathcal{I}_C$ is the ideal sheaf associated to C. This sequence will be useful when we start working in the derived category.

6.2. Stable pairs in the derived category

We will now illustrate how stable pairs can be viewed as elements in the derived category, and how we deform the stable pairs in this setting.

A stable pair $(\mathcal{F}, s)$ can be viewed as a complex

$I^\bullet = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}]$ located in degrees 0 and 1 in $\mathcal{D}^b(X)$.

It can be shown [**PT09**, p.11] that two stable pairs are isomorphic precisely when their corresponding complexes are quasi-isomorphic.

We will make use of two important exact triangles associated to a stable pair $(\mathcal{F}, s)$.

Claim 6.2. A stable pair $(\mathcal{F}, s)$ admits the following exact triangles:

1. $\mathcal{F}[-1] \longrightarrow I^\bullet \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow \dots$
2. $\mathcal{I}_C \longrightarrow I^\bullet \longrightarrow Q[-1] \longrightarrow \mathcal{I}_C[1] \longrightarrow \dots$

Proof. 1. The result follows since $I^\bullet[1]$ is the mapping cone corresponding to the morphism of complexes $\mathcal{O}_X \xrightarrow{s} F$.

For 2, we note that the mapping cone of $\mathcal{I}_C \longrightarrow I^\bullet$ is the complex

$$[\overset{-1}{\mathcal{I}_C} \longrightarrow \overset{0}{\mathcal{O}_X} \xrightarrow{s} \overset{1}{\mathcal{F}}],$$

where degrees are indicated above the terms in the complex.

The result follows from the quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_C \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r, "s"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
& 0 \arrow[r] & 0 \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

arising from the exact sequence (13) (compare this to Example 5.1). $\square$

To deform the stable pairs in the derived category, we will use the interpretation in Lemma 6.3 below. For this, we note that the determinant of a complex of coherent sheaves can be defined as the alternately dual tensor product of the determinants of the terms in the complex.

LEMMA 6.3. *A stable pair* $I^\bullet = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}]$ *defines an object in* $\mathcal{D}^b(X)$ *with trivial determinant.*

PROOF. Since $\operatorname{codim}(\mathcal{F}) = 2$, the determinant $\det(\mathcal{F})$ is trivial. So $\det(I^\bullet) = \det(\mathcal{O}_X) \otimes \mathcal{O}_X^\vee \cong \mathcal{O}_X$ is trivial. □

Before deforming the complexes associated to the stable pairs, we need to develop the deformation theory of complexes in $\mathcal{D}^b(X)$. We will see many similarities to the deformation theory for coherent sheaves.

Let $K^\bullet$ be a complex on X. As a replacement for the flatness condition in the case of coherent sheaves, we require that the complex is *perfect.* This means that the complex is quasi-isomorphic to a complex in $\mathcal{D}^b(X)$ of locally free sheaves. Since locally free sheaves are flat, perfection of $K^\bullet$ implies that the functor $K^\bullet \otimes^{\mathbf{L}}$ preserves exact triangles in $\mathcal{D}^b(X)$.

We define a *deformation* of a perfect complex $K^\bullet$ on X over a base extension $\operatorname{Spec}(k) \subset B$ to be a perfect complex $L^\bullet$ on $X \times B$ equipped with a fixed isomorphism $\mathbf{L}\iota^* L^\bullet \cong K^\bullet$, where $\iota\colon X \hookrightarrow X \times B$ is the inclusion defined by the base extension.

To illustrate the similarity to the deformation theory for coherent sheaves, we consider deformations of a perfect complex $K^\bullet$ over the dual numbers D.

Taking the derived tensor product of the short exact sequence (6)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X \times D} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

with $L^\bullet$ gives the distinguished triangle

$$\mathcal{O}_X \otimes^{\mathbf{L}} L^\bullet \longrightarrow L^\bullet \longrightarrow \mathcal{O}_X \otimes^{\mathbf{L}} L^\bullet \longrightarrow \mathcal{O}_X \otimes^{\mathbf{L}} L^\bullet[1] \longrightarrow \dots$$

The terms $\mathcal{O}_X \otimes^{\mathbf{L}} L^\bullet$ are just the derived restrictions of $L^\bullet$, and so we can view $L^\bullet$ as an element of $\operatorname{Hom}_{\mathcal{D}^b(X)}(K^\bullet, K^\bullet[1]) = \operatorname{Ext}^1(K^\bullet, K^\bullet)$. Analogous to the deformations of coherent sheaves, we obtain the deformation space $\operatorname{Ext}^1_0(I^\bullet, I^\bullet)$ for the complex $I^\bullet = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}]$ associated to our stable pair, where Ext_0 denotes the splitting associated to the kernel of a certain trace map. Similarly, one can prove that the obstruction space is $\operatorname{Ext}^2_0(I^\bullet, I^\bullet)$.

In [**PT09**], Pandharipande and Thomas prove that the deformations of stable pairs in the derived category with fixed determinant correspond precisely to deformations of stable pairs in the following sense:

A *family of stable* pairs over a base B is a pair (not assumed to be stable) $\mathcal{O}_{X \times B} \xrightarrow{s} \mathcal{F}'$, where $\mathcal{F}'$ is a coherent sheaf on $X \times B$, flat over B, such that the restrictions $(\mathcal{F}'_b, s_b)$ form stable pairs on X for each closed $b \in B$.

Theorem 2.7 in [**PT09**] states that every deformation of $I^\bullet$ over B with trivial determinant is quasi-isomorphic to a complex $[\mathcal{O}_{X \times B} \xrightarrow{s} \mathcal{F}']$ corresponding to a family of stable pairs, where $\mathcal{F}'$ is a deformation of $\mathcal{F}$.

Using an appropriate obstruction theory, we obtain a virtual fundamental class $[P_n(X, \beta)]^{vir}$ for the moduli space of stable pairs. We can thus define the PT invariants:

DEFINITION 6.4. For a given curve class (β, n) on a Calabi-Yau 3-fold X, the corresponding *Pandharipande-Thomas invariant* is defined as $P_{\beta,n} = \deg([P_n(X,\beta)]^{vir})$.

As before, we can interpret these invariants as weighted Euler characteristics;

$$P_{\beta,n} = \chi(P_n(X,\beta),\nu).$$

Moreover, the virtual fundamental class is deformation invariant in the sense of [**PT09**, Thm 2.15].

In the coming chapter, we will relate the invariants $I_{\beta,n}$ and $P_{\beta,n}$.

CHAPTER 7

Relating the DT- and PT invariants

> *Il y a autant de "mathématiques" qu'il y a de mathématiciens*
>
> A. Grothendieck, *Récoltes et Semailles*

To relate the two curve counting invariants, we will find it useful to work with several values of n at the same time, still keeping the curve class β fixed. To do this efficiently, we form the following generating series for the DT- resp. PT invariants:

$$I_\beta(q) = \sum_{n \in \mathbb{Z}} I_{\beta,n} q^n \text{ resp. } P_\beta(q) = \sum_{n \in \mathbb{Z}} P_{\beta,n} q^n.$$

From these series, we can try to find relations among the coefficients.

We will describe a way of interpreting objects in $I_n(X,\beta)$ and $P_n(X,\beta)$ that will make it possible to relate these moduli spaces, following Bridgeland's approach in [**Br11**].

7.1. Relating the objects of $I_n(X,\beta)$ and $P_n(X,\beta)$

Consider the subcategory $\mathcal{A} = \mathrm{Coh}(X) \subset \mathcal{D}^b(X)$. We will interpret elements of $I_n(X,\beta)$ as follows:

An ideal sheaf $\mathcal{I}_Z \in I_n(X,\beta)$ corresponds to a surjective map $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$. Hence, the objects in $I_n(X,\beta)$ can be seen as surjections $\mathcal{O}_X \twoheadrightarrow E$ for sheaves $E \in \mathcal{A}$ of dimension ≤ 1. By considering the subsheaf T of an object $E \in \mathcal{A}$ consisting of the elements with 0-dimensional support, we can form an exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow S \longrightarrow 0$$

where

$$T \in \mathcal{T} = \{E \in \mathcal{A} \mid \dim(E) = 0\}$$

and

$$S \in \mathcal{S} = \{E \in \mathcal{A} \mid \mathrm{Hom}(T,E) = 0 \text{ for all } T \in \mathcal{T}\}.$$

From the pair $(\mathcal{T},\mathcal{S})$, we can (the method is called *tilting*, see [**HRS96**]), obtain a new subcategory

$$\mathcal{A}^\# = \{E \in \mathcal{D}^b(X) \mid H^0(E) \in \mathcal{S}, H^1(E) \in \mathcal{T}, H^i(E) = 0 \text{ for } i \notin \{0,1\}\},$$

where $H^i(E)$ denotes the complex consisting of the i^{th} cohomology of the complex E, situated in degree 0.

It turns out that a complex $I^\bullet = [\mathcal{O}_X \xrightarrow{s} \mathcal{F}]$ associated to a stable pair can be viewed as an object in $\mathcal{A}^\#$. Indeed, we first note that the cokernel $H^1(I^\bullet) = \mathrm{coker}(s)$ has dimension 0 and so lies in $\mathcal{T}$. The kernel, on the other hand, is the ideal sheaf $\mathcal{I}_C$ of a curve with no embedded points. A nonzero morphism from T to

$\mathcal{I}_C$ where $T \in T$ gives a nonzero subsheaf of $\mathcal{O}_X$ supported in dimension 0, which is not possible since X is integral. This means that $\mathcal{I}_C$ lies in $\mathcal{S}$. Purity of the sheaf $\mathcal{F}$ in a stable pair $(\mathcal{F}, s)$ also gives us $\mathcal{F} \in \mathcal{S}$.

While the elements of $I_n(X, \beta)$ are surjections $\mathcal{O}_X \twoheadrightarrow E$ in $\mathcal{A}$, the elements $(\mathcal{F}, s)$ of $P_n(X, \beta)$ are instead surjections in $\mathcal{A}^{\#}$, meaning that there is an exact triangle of the form

$$J \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{F} \longrightarrow J[1] \longrightarrow \dots$$

for $J \in \mathcal{A}^{\#}$. Indeed, the sequence in 6.2.1 shows that we can pick $J = I^{\bullet}$.

Conversely, given an exact triangle

$$J \longrightarrow \mathcal{O}_X \xrightarrow{s} E \longrightarrow J[1] \longrightarrow \dots$$

for $E, J \in \mathcal{A}^{\#}$, the long exact sequence in cohomology (12) is

$$0 \longrightarrow H^0(J) \longrightarrow \mathcal{O}_X \longrightarrow H^0(E) \longrightarrow H^1(J) \longrightarrow 0 \longrightarrow H^1(E) \longrightarrow 0.$$

This means that $E = H^0(E)$ lies in $\mathcal{A} \cap \mathcal{A}^{\#} = \mathcal{S}$ and $\operatorname{coker}(s) = H^1(J)$ lies in $\mathcal{T}$. Thus, if the sheaf E is 1-dimensional, we precisely recover a stable pair $\mathcal{O}_X \longrightarrow E$.

7.2. Hall algebras

> *I wonder if any one out in this*
> *wilderness counts his stacks, and*
> *compares them with his neighbour's?*
> S. Lagerlöf, *Further Adventures of Nils*

Bridgeland's method for relating the invariants in [**Br11**] (another proof by Toda can be found in [**To14**]) is to work in the *motivic Hall algebra*. To motivate this, we first describe the *finitary Hall algebra*. The construction requires certain finiteness constraints on the category (see [**Br16**, p.6]), but is easy to understand and captures essential ideas. We choose an explicit category for notational simplicity.

Definition 7.1. Let k be a finite field, and R a k-algebra which is finite-dimensional as a k-vector space (e.g. $R = k[X]/(X^n)$, k^n, $\mathrm{Mat}_{n\times n}(k)$). Let $\mathcal{B}$ be the category of finite-dimensional R-modules. The *finitary Hall algebra* $\mathrm{Hall}(\mathcal{B})$ is the set of complex functions

$$\{f\colon (\mathrm{Obj}(\mathcal{B})/\cong) \to \mathbb{C}\}$$

on the isomorphism classes of modules in $\mathcal{B}$, equipped with a convolution product

$$(f_1 * f_2)(B) = \sum_{A \subset B} f_1(A) \cdot f_2(B/A).$$

For instance, if we let S_1 and S_2 be two subsets of $\mathcal{B}$ and δ_{S_1}, δ_{S_2} be indicator functions for these sets, the product $(\delta_{S_1} * \delta_{S_2})(B)$ counts the number of ways a module $B \in \mathcal{B}$ can be written as an extension

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with $A \in S_1$ and $C \in S_2$, up to the equivalence induced by equal images of A in B. As an example, let $R = \mathbb{F}_q$ and let δ_n denote the indicator function for the isomorphism class of n-dimensional vector spaces over R. For an $(n+m)$-dimensional vector space V, the value $(\delta_n * \delta_m)(V)$ is the number of n-dimensional subspaces of V.

More generally, for subsets $S_1, \dots, S_n \subset \mathcal{B}$, the product $(\delta_{S_1} * \delta_{S_2} * \cdots * \delta_{S_n})(B)$ counts the number of ways an object B can be filtered as

$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n = B \tag{14}$$

with $B_i/B_{i-1} \in S_i$ for $i = 1, \dots, n$.

This might be of interest when counting the number of elements in a moduli space whose elements can be classified by filtrations as in (14) with successive quotients lying in specified subcategories.

We now introduce the motivic Hall algebra. For us, the category of interest will be $\mathcal{A} = \mathrm{Coh}(X)$. To obtain desirable properties of the motivic Hall algebra, we further restrict to the subcategory $\mathcal{C} \subset \mathcal{A}$ consisting of sheaves supported in dimension ≤ 1. The objects $K(\mathrm{St}/\mathcal{C})$ of interest will be *Artin stacks* $\mathcal{M}$ over $\mathbb{C}$ equipped with a morphism $\mathcal{M} \xrightarrow{f} \mathcal{C}$. The definition of an Artin stack is omitted here; a scheme is just a special case of an Artin stack (see instead [**Fa01**]). When using the term *stacks* from now on, we will refer to Artin stacks which are locally of finite type with *affine stabilizers* (see [**Br10**, p.10]).

To construct an algebra from $K(\mathrm{St}/\mathcal{C})$, we need both a group structure, multiplication and a ring of scalars. For the scalars, we consider the *Grothendieck ring* $K(\mathrm{St}/\mathbb{C})$ of stacks $[\mathcal{X}] = [\mathcal{X} \xrightarrow{f} \mathbb{C}]$ over $\mathbb{C}$. The objects are subject to the following relations:

1. $[\mathcal{X}] = [\mathcal{Y}]$ whenever there is a morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ which is representable (informally, the fibers over schemes are schemes [**Fa01**, p.7]) and induces a categorical equivalence on the $\mathbb{C}$-points $\mathcal{X}(\mathbb{C}) \longrightarrow \mathcal{Y}(\mathbb{C})$ (for schemes, this just means that the morphism f induces a bijection of $\mathbb{C}$-points as sets).

2. $[\mathcal{X}_1] = [\mathcal{X}_2]$ whenever $f_i : \mathcal{X}_i \longrightarrow \mathcal{Y}$ are *Zariski fibrations* with the same fibers. The morphisms being Zariski fibrations means that, pulling back the morphisms to a scheme S, we can cover S by open sets U_i such that there are commutative diagrams

$$\begin{tikzcd}
(f \times_Y \mathrm{id}_S)^{-1}(U_i) \arrow[rr, "\cong"] \arrow[dr, "f \times_Y \mathrm{id}_S"'] & & U_i \times_{\mathbb{C}} F_i \arrow[dl, "\mathrm{pr}_1"] \\
& U_i. &
\end{tikzcd}$$

Addition is given by the relation $[\mathcal{X}_1] + [\mathcal{X}_2] = [\mathcal{X}_1 \sqcup \mathcal{X}_1]$, and multiplication by the fiber product $[\mathcal{X}_1] \cdot [\mathcal{X}_2] = [\mathcal{X}_1 \times \mathcal{X}_2]$.

As an example, we can deduce the "scissor relation"

$$[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} \backslash \mathcal{Y}]$$

whenever $\mathcal{Y} \subset \mathcal{X}$ is a closed substack. Indeed, the morphism $\mathcal{Y} \sqcup \mathcal{X} \backslash \mathcal{Y} \longrightarrow \mathcal{X}$ given by inclusion of sets is a geometric bijection.

We now give the remaining structure of the algebra.

Definition 7.2. The *motivic Hall algebra* $\mathrm{H}(\mathcal{C})$ consists of objects $[\mathcal{M} \xrightarrow{f} \mathcal{C}]$ where $\mathcal{M}$ is a stack over $\mathbb{C}$, subject to equivalence relations as in [**Br11**, pp.14-15] generalizing those of $K(\mathrm{St}/\mathbb{C})$. Addition is given by

$$[\mathcal{M}_1 \longrightarrow \mathcal{C}] + [\mathcal{M}_1 \longrightarrow \mathcal{C}] = [\mathcal{M}_1 \sqcup \mathcal{M}_2 \longrightarrow \mathcal{C}],$$

and the module structure over $K(\mathrm{St}/\mathbb{C})$ by

$$[\mathcal{X}] \cdot [\mathcal{M} \longrightarrow \mathcal{C}] = [\mathcal{X} \times_{\mathbb{C}} \mathcal{M} \longrightarrow \mathcal{C}].$$

For us, the most important aspect of the algebra will be its product, given as follows: Consider the stack $\mathrm{SES}(\mathcal{C})$ of short exact sequences of objects in $\mathcal{C}$. For two stacks $[\mathcal{M}_1 \xrightarrow{f_1} \mathcal{C}]$ and $[\mathcal{M}_1 \xrightarrow{f_1} \mathcal{C}]$, consider the diagram

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "h"] \arrow[d] & \mathrm{SES}(\mathcal{C}) \arrow[r, "b"] \arrow[d, "(a,c)"] & \mathcal{C} \\
\mathcal{M}_1 \times \mathcal{M}_2 \arrow[r, "f_1 \times f_2"] & \mathcal{C} \times \mathcal{C} &
\end{tikzcd}$$

where the maps b resp. (a, c) sends a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \;\in \mathrm{SES}(\mathcal{C})$$

to its middle term B resp. outer terms (A, C), and $\mathcal{N}$ is the fiber product making the square Cartesian. The product, denoted $[\mathcal{M}_1 \xrightarrow{f_1} \mathcal{C}] * [\mathcal{M}_1 \xrightarrow{f_1} \mathcal{C}]$, is defined to be the stack $[\mathcal{N} \xrightarrow{b \circ h} \mathcal{C}]$. Informally, $\mathcal{N}$ is the stack of extensions of elements in $\mathcal{M}_2$ by elements in $\mathcal{M}_1$ (respecting the morphisms to $\mathcal{C}$ and with suitable equivalence relations imposed). Compare this to the product associated to the finitary Hall algebra.

The motivic Hall algebra is not commutative, but has the multiplicative identity element $[0 \hookrightarrow \mathcal{C}]$.

To place the Hilbert scheme and the stable pairs in this setting, we introduce the stacks (which are in fact schemes) $\mathcal{H}$ and $\mathcal{P}$ parameterizing maps $\{\mathcal{O}_X \longrightarrow E\}$ for $E \in \mathcal{C}$ resp. $\mathcal{C} \cap \mathcal{A}^{\#} = \mathcal{C} \cap \mathcal{S}$ which are surjective in $\mathcal{A}$ resp. $\mathcal{A}^{\#}$. The connected components of $\mathcal{H}$ are precisely the schemes $I_n(X, \beta)$, whereas for $\mathcal{P}$ they are $P_n(X, \beta)$. For an open substack $\mathcal{M} \subset \mathcal{C}$, we denote the element $[\mathcal{M} \hookrightarrow \mathcal{C}]$ by $\delta_{\mathcal{M}}$. Moreover, we let $\mathcal{M}^{\mathcal{O}}$ denote the stack of elements of $\mathcal{M}$ equipped with a section. Composing the map from $\mathcal{M}^{\mathcal{O}}$ to $\mathcal{M}$ obtained by forgetting the section with the inclusion map from $\mathcal{M}$ to $\mathcal{C}$, we denote the resulting element $[\mathcal{M}^{\mathcal{O}} \longrightarrow \mathcal{C}] \in H(\mathcal{C})$ by $\delta^{\mathcal{O}}_{\mathcal{M}}$.

We will mostly be concerned with the elements $\delta_{\mathcal{H}}$, $\delta_{\mathcal{P}}$, $\delta_{\mathcal{T}}$ and $\delta_{\mathcal{S}_1}$ with $\mathcal{S}_1 := \mathcal{C} \cap \mathcal{S}$.

The following Hall algebra identities will help us relate $\mathcal{H}$ and $\mathcal{P}$:

$$1.\ \delta_{\mathcal{H}} * \delta_{\mathcal{C}} = \delta^{\mathcal{O}}_{\mathcal{C}},\ 2.\ \delta_{\mathcal{P}} * \delta_{\mathcal{S}_1} = \delta^{\mathcal{O}}_{\mathcal{S}_1},\ 3.\ \delta_{\mathcal{T}} * \delta_{\mathcal{S}_1} = \delta_{\mathcal{C}},\ \text{and}\ 4.\ \delta^{\mathcal{O}}_{\mathcal{T}} * \delta^{\mathcal{O}}_{\mathcal{S}_1} = \delta^{\mathcal{O}}_{\mathcal{C}}.$$

Moreover, letting $\mathcal{H}_0$ be the substack of $\mathcal{H}$ parameterizing maps $\{\mathcal{O}_X \longrightarrow E\}$ where E has 0-dimensional support, we have the identity

$$5.\ \delta_{\mathcal{H}_0} * \delta_{\mathcal{T}} = \delta^{\mathcal{O}}_{\mathcal{T}}.$$

We will explain the intuitive meaning of these identities, without going into details of the proofs (details can be found in [**Br11**]).

For the first identity, consider a diagram

$$\begin{tikzcd}
 & \mathcal{O}_X \arrow[d, two heads] & & & \\
0 \arrow[r] & E \arrow[r, "g"] & F \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

where $\{\mathcal{O}_X \overset{f}{\twoheadrightarrow} E\} \in \mathcal{H}$ and $G \in \mathcal{C}$ are given. A sheaf $F \in \mathcal{C}$ fitting into this sequence determines an element $\{\mathcal{O}_X \xrightarrow{g \circ f} F\}$ in $\mathcal{C}^{\mathcal{O}}$.

Conversely, given an element $\{\mathcal{O}_X \xrightarrow{s} F\} \in \mathcal{C}^{\mathcal{O}}$, we obtain a sequence

$$\begin{tikzcd}
 & \mathcal{O}_X \arrow[d, two heads] \arrow[dr, dashed, "s"] & & & \\
0 \arrow[r] & \mathrm{im}(s) \arrow[r] & F \arrow[r] & \mathrm{coker}(s) \arrow[r] & 0
\end{tikzcd}$$

with $\{\mathcal{O}_X \twoheadrightarrow \mathrm{im}(s)\} \in \mathcal{H}$ and $\mathrm{coker}(s) \in \mathcal{C}$.

Here, we implicitly use the fact that, for a short exact sequence of sheaves

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, we have (topologically) $\mathrm{Supp}(B) = \mathrm{Supp}(A) \cup \mathrm{Supp}(C)$. Therefore, given that two of the terms in the sequence belong to $\mathcal{C}$, the third term does as well.

Thus, we can see elements of $\mathcal{C}^{\mathcal{O}}$ as "extensions" by elements of $\mathcal{C}$ with elements of $\mathcal{H}$, motivating the relation $\delta_{\mathcal{H}} * \delta_{\mathcal{C}} = \delta_{\mathcal{C}}^{\mathcal{O}}$.

Since elements in $\mathcal{H}_0$ are precisely surjective maps $\{\mathcal{O}_X \twoheadrightarrow T\}$ with $T \in \mathcal{T}$, identity 5 is motivated analogously.

For 3, we recall that any element $E \in \mathcal{C}$ fits into a sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow S \longrightarrow 0$$

with $T \in \mathcal{T}$ and $S \in \mathcal{S}_1$.

Since sheaves in $\mathcal{T}$ are supported in dimension 0, the cohomology $H^1(X, T)$ vanishes by Grothendieck's vanishing theorem. This means that the sequence of global sections

$$0 \longrightarrow \Gamma(X, T) \longrightarrow \Gamma(X, E) \longrightarrow \Gamma(X, S) \longrightarrow 0$$

is exact. Combining this with identity 3 motivates identity 4.

Lastly, the identity 2 will follow from the earlier mentioned fact that an element $\{\mathcal{O}_X \longrightarrow E\} \in \mathcal{P}$ gives an element $E \in \mathcal{S}_1$. The rest of the argument is similar to the motivation for identity 1.

We now wish to combine the identities 1-5 to give a relation between the elements $\delta_{\mathcal{H}}$ and $\delta_{\mathcal{P}}$. For this, we note that the identities 1, 2 and 4 give us

$$\delta_{\mathcal{T}}^{\mathcal{O}} * \delta_{\mathcal{P}} * \delta_{\mathcal{S}_1} = \delta_{\mathcal{T}}^{\mathcal{O}} * \delta_{\mathcal{S}_1}^{\mathcal{O}} = \delta_{\mathcal{C}}^{\mathcal{O}} = \delta_{\mathcal{H}} * \delta_{\mathcal{C}},$$

which together with identities 3 and 5 give

$$\delta_{\mathcal{H}_0} * \delta_{\mathcal{T}} * \delta_{\mathcal{P}} * \delta_{\mathcal{S}_1} = \delta_{\mathcal{H}} * \delta_{\mathcal{T}} * \delta_{\mathcal{S}_1}.$$

In [**Br11**, p.29], by considering smaller subcategories of $\mathcal{C}$ and $\mathcal{S}_1$ induced by a stability filtration (see [**Br11**, p.25]), Bridgeland proves that the term $\delta_{\mathcal{S}_1}$ cancels from both sides. Thus, we have a useful identity

$$\delta_{\mathcal{H}_0} * \delta_{\mathcal{T}} * \delta_{\mathcal{P}} = \delta_{\mathcal{H}} * \delta_{\mathcal{T}} \tag{15}$$

relating $\delta_{\mathcal{P}}$ and $\delta_{\mathcal{H}}$. The meaning of this identity can be interpreted through filtrations as in (14).

7.3. The DT/PT identity

We will see how the identity (15) can be broken into pieces to relate separate classes (β, n), simultaneously encoding the virtual count.

We first restrict to the subalgebra $H_{\mathrm{reg}}(\mathcal{C}) \subset H(\mathcal{C})$ consisting of elements $[\mathcal{M} \xrightarrow{f} \mathcal{C}]$ where $\mathcal{M}$ is a scheme. We consider $H_{\mathrm{reg}}(\mathcal{C})$ as an algebra over the subring of $K(\mathrm{St}/\mathbb{C})$ consisting of varieties over $\mathbb{C}$ with the classes $[\mathbb{A}^1]$ and $[\mathbb{P}^n]$ for $n \geq 1$ inverted.

Next, we consider the quotient algebra $H_{\mathrm{sc}}(\mathcal{C}) := H_{\mathrm{reg}}(\mathcal{C})/([\mathbb{C}^*]H_{\mathrm{reg}}(\mathcal{C}))$ (sc stands for "semi-classical").

Let $K_0(\mathcal{C})$ be the *Grothendieck group* of coherent sheaves of dimension ≤ 1 on a CY 3-fold X, defined as follows: Elements are $\mathbb{Z}$-linear combinations of symbols $[\mathcal{F}]$ for $\mathcal{F} \in \mathcal{C}$, subject to the relation $[\mathcal{F}'] + [\mathcal{F}''] = [\mathcal{F}]$ for every short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

The Chern character gives a homomorphism

$$(\mathrm{ch}_2, \mathrm{ch}_3)\colon K_0(\mathcal{C}) \longrightarrow H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$$

mapping a sheaf $[\mathcal{F}]$ to its class (β, n), where $\beta = [\mathrm{Supp}(\mathcal{F})]$ and $n = \chi(\mathcal{F})$. We denote the image, which is in fact a lattice [**Br11**, p.4], by N. There is an effective cone $\Delta = \{(\beta, n) \in N | \beta > 0 \text{ or } \beta = 0 \text{ and } n \geq 0\}$ being the image of the classes of sheaves.

We define an algebra $\mathbb{C}[\Delta] := \bigoplus_{(\beta,n)\in\Delta} \mathbb{C}x^{(\beta,n)}$ by the product

$$x^{(\beta_1,n_1)} \cdot x^{(\beta_2,n_2)} = x^{(\beta_1+\beta_2,n_1+n_2)}.$$

There is an algebra homomorphism, referred to as an *integration map*, given by $I\colon H_{\mathrm{sc}}(\mathcal{C}) \longrightarrow \mathbb{C}[\Delta]$ such that

$$I([\mathcal{M} \xrightarrow{f} \mathcal{C}]) = \chi(\mathcal{M}_{(\beta,n)}, f^*(\nu_{\mathcal{C}}))x^{(\beta,n)},$$

where $\mathcal{M}_{(\beta,n)}$ is the subscheme of $\mathcal{M}$ consisting of sheaves in the class (β, n) and $\nu_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{Z}$ is the Behrend function.

In particular, using properties of the Behrend function as in [**Br11**, p.9] and the interpretations of the DT/PT invariants as weighted Euler characteristics, we obtain $I(\delta_{\mathcal{H}}) = I_\beta(-q)x^{(\beta,0)}$ and $I(\delta_{\mathcal{P}}) = P_\beta(-q)x^{(\beta,0)}$ where $q = x^{(0,1)}$.

Applying the integration map to the identity (15), noting that $\mathbb{C}[\Delta]$ is commutative, we obtain

$$I(\delta_{\mathcal{H}_0}) \cdot I(\delta_{\mathcal{P}}) = I(\delta_{\mathcal{H}}).$$

This precisely states that we have a relation between the generating series of the DT- and PT invariants:

$$I_0(q)P_\beta(q) = I_\beta(q).$$

Comparing the coefficients, we deduce the following beautiful identity, as conjectured in [**PT09**]:

$$\sum_{m\in\mathbb{Z}} P_{n-m,\beta} I_{m,0} = I_{n,\beta}.$$

Summary and future directions

We now summarize the ideas described in this essay. We started with the aim of counting curves on Calabi-Yau 3-folds, knowing that the number of curves of a given class (β, n) on a CY 3-fold is expected to be finite. We studied two compactifications of the moduli space of curves of the class (β, n) by giving geometrical interpretations and considering the deformation theory of the objects they parameterize. We explained why it is beneficial to introduce a virtual fundamental class to obtain counts which are invariant upon deforming the 3-fold. We gave an idea of the procedure for constructing virtual fundamental classes associated to the moduli spaces in question. We further introduced the derived category of coherent sheaves and indicated how to deform stable pairs in this setting. Finally, we used the motivic Hall algebra to find a correspondence between the DT- and PT invariants involving their generating series.

We did not, however, introduce the concept of stability, which is essential among other things for defining the higher rank DT invariants counting stable coherent sheaves. While many results have been proven using Giseker stability, it would be favorable to work with stability conditions in the derived category as introduced by Bridgeland in [**Br07**], e.g. to study wall-crossing as in [**KS13**], [**JS12**]. Currently, there are very few known stability conditions (e.g. [**Li19**] and [**BMS16**]) for CY 3-folds, which means there is more research to be done before we can study wall-crossing efficiently. A certain Bridgeland stability condition would give an interpretation of the relation between DT- and PT- invariants as a wall-crossing in the derived category (see [**PT09**]).

The DT- and PT invariants both play a role in the OSV-conjecture, relating black hole thermodynamics to topological string theory through partition functions. The mathematical analogue of this conjecture, as formulated by Denef and Moore in [**DM11**] and refined by Toda in [**To13**], is a conjectured relation involving the DT/PT invariants and certain higher rank DT invariants counting torsion sheaves supported on an ample divisor. The relation has been proven under certain hypotheses in [**To13**] and [**Fe23**].

In short, there is much algebraic geometry to be carried out before we can gain a better grasp of our universe.

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