Chiral Symmetry Breaking

Abstract

The aim of this essay is to give an overview of the chiral symmetry in the massless QCD limit and discuss the current algebra and field configuration aspects of the effective chiral Lagrangian.

QCD is a quantum field theory that describes the strong interactions of quarks and gluons. It exhibits a confinement phenomenon because of which we do not observe individual quarks. Instead, we see hadron particles such as mesons and baryons which contain an even and odd number of quarks respectively. The lightest mesons in the hadronic spectrum are known as pions.

The physics of pions is largely constrained by the chiral symmetry. This symmetry is broken by an order parameter called chiral condensate, and by Goldstone's Theorem, pions are viewed as massless Goldstone bosons (although they have a mass of 100 MeV). This perspective allows us to write an effective field theory for pions, called the chiral Lagrangian. From this Lagrangian, one can investigate pion interaction processes through the current algebra approach. However, it turns out that this effective Lagrangian does not capture all decay modes, such as the anomalous neutral pion decay mode $\pi^0 \to \gamma + \gamma$. Nevertheless, one can consider certain extended field configurations to the chiral Lagrangian, such as introducing the 5-dimensional Wess-Zumino action, to account for anomalous QCD effects in low-energy processes.

We start by introducing chiral symmetry in QCD and discussing how the chiral/ABJ anomaly rises in the massless QCD limit in Chapter 1. We proceed by investigating current algebra aspects of the chiral lagrangian and how it can be linked to the chiral anomaly in the context of neutral pion decay. In Chapter 2 we investigate the topological aspects of the ABJ anomaly. Lastly, in Chapter 3 we delve into further field configurations of the chiral Lagrangian and discuss the mathematical framework of the Wess-Zumino-Witten effective action term, and how its gauge-invariant version leads us again to the ABJ anomaly.

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Chapter 1

Chiral Symmetry

1.1 QCD Lagrangian

Let us now consider the QCD Lagrange density for quark q and gluon G_μ fields [16]

$$\mathcal{L}_{\text{QCD}}(q,\bar{q},G_{\mu\nu}) = -\frac{1}{2}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) + \sum_{i=1}^{N_f}\bar{q}_i(i\not{D} - M)q_i \tag{1.1}$$

where in Dirac notation the covariant derivative is

$$\not{D} = \gamma^\mu\left(\partial_\mu + g_s G_\mu(x)\right). \tag{1.2}$$

Here the summation index $i = 1,\ldots,N_f$ labels the species of quarks and is known as the flavour index. Each quark spinor q component carries a colour and a spinor index but both of these indices are suppressed in (1.1). It should be noted that γ^μ here are 4×4 matrices composed from 2×2 Pauli spin matrices σ^i

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}_{4\times 4}, \quad \begin{matrix} \sigma^\mu = \left(1_{2\times 2}, \quad {\sigma^i}_{2\times 2}\right) \\ \bar{\sigma}^\mu = \left(1_{2\times 2}, \quad -{\sigma^i}_{2\times 2}\right) \end{matrix}, \quad i = 1,2,3 \tag{1.3}$$

and they generate Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}1_{4\times 4}$. Let us also define the chirality γ^5 matrix

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}_{4\times 4} = {\gamma^\dagger}_5, \quad {\gamma_5}^2 = 1_{4\times 4}. \tag{1.4}$$

We may decompose the 4-component Dirac quark fields into left-handed and right-handed 2-component Weyl spinor fields

$$q = \begin{pmatrix} q_L \\ q_R \end{pmatrix} \tag{1.5}$$

and we also define the projection operators in a chiral basis [7]

$$P_L = \frac{1}{2}\left(1 - \gamma^5\right) = P^{\dagger}{}_L, \quad P_R = \frac{1}{2}\left(1 + \gamma^5\right) = P^{\dagger}{}_R. \tag{1.6}$$

The projection operators allow writing the left-handed and right-handed quark components as

$$q_L = P_L q = \frac{1}{2}\left(1 - \gamma^5\right) q, \quad q_R = P_R q = P_R = \frac{1}{2}\left(1 + \gamma^5\right) q, \quad q = q_L + q_R. \tag{1.7}$$

Hence, using this notation we can write the QCD Lagrangian in terms of Weyl spinors

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + \sum_{i=1}^{N_f}\left(i\bar{q}_{L_i}\not{D}q_{L_i} + i\bar{q}_{R_i}\not{D}q_{R_i} - \bar{q}_{L_i}Mq_{R_i} - \bar{q}_{R_i}Mq_{L_i}\right). \tag{1.8}$$

1.2 Chiral Symmetry

Without the mass terms, the QCD Lagrangian (1.8) is invariant under separate $U(N_f)_L$ and $U(N_f)_R$ transformations of the left-handed and right-handed components in flavour space

$$q_{L_i} \mapsto L_{ij}q_{L_j} \quad L_{ij} \in U(N_f)_L \tag{1.9}$$

$$q_{R_i} \mapsto R_{ij}q_{R_j} \quad R_{ij} \in U(N_f)_R \tag{1.10}$$

and hence the massless $\mathcal{L}_{\text{QCD}}$ has a global symmetry

$$G_F = U(N_f)_L \otimes U(N_f)_R. \tag{1.11}$$

We can write down two more $U(1)$ symmetries for the massless QCD Lagrangian

$$Vector \quad U(1)_V: \quad q_{L_i} \mapsto e^{i\alpha}q_{L_i}, \quad q_{R_i} \mapsto e^{i\alpha}q_{R_i} \tag{1.12}$$

$$Axial \quad U(1)_A: \quad q_{L_i} \mapsto e^{i\beta}q_{L_i}, \quad q_{R_i} \mapsto e^{-i\beta}q_{R_i}. \tag{1.13}$$

The $U(1)_V$ symmetry survives in the quantum theory and in the context of QCD the associated quantity of this symmetry counts the number of baryons. The other $U(1)_A$ symmetry is more subtle. Although it is a symmetry in the classical field theory, in the quantum theory it suffers the *ABJ anomaly* [1, 6, 4].

1.3 The ABJ Anomaly

To see how the ABJ anomaly arises, we will adopt Kazuo Fujikawa's interpretation [11] of the anomaly. The intuitive interpretation is that the anomaly occurs when the symmetries of the classical action are not the symmetries of the functional measure in the quantum path integral.

For simplicity, we start with an illustrative anomaly computation for an Abelian QED theory, and the final result can be generalised to non-Abelian theories (e.g. QCD).

The QED Lagrangian for fermionic field ψ is

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_\mu D^\mu - m)\psi \tag{1.14}$$

where spinor ψ can be decomposed to both left-handed and right-handed components using the same arguments as for QCD discussed above, and in the massless limit, one can write

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}_L \not{D}\psi_L + i\bar{\psi}_R \not{D}\psi_R. \tag{1.15}$$

Written this way, the massless QED theory is invariant under the following symmetry transformations

$$\begin{array}{rcl} \psi \mapsto e^{i\alpha}\psi & & \psi_L \mapsto e^{i(\alpha-\beta)}\psi_L \\ & \Rightarrow & \\ \psi \mapsto e^{i\beta\gamma^5}\psi & & \psi_R \mapsto e^{i(\alpha+\beta)}\psi_R \end{array} \tag{1.16}$$

and using the Noether's theorem, one finds the axial and vector currents as

$$\begin{array}{rl} Axial: & J_A{}^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \\ Vector: & J_V{}^\mu = \bar{\psi}\gamma^\mu\psi \end{array}. \tag{1.17}$$

In the massless limit, both currents in the classical action are conserved

$$\partial_\mu J_V{}^\mu = \quad \partial_\mu J_A{}^\mu = 0 \tag{1.18}$$

while in the quantum path integral, as we mentioned, only the Vector current is conserved.

To compute this effect, consider the following path integral

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}Ae^{i\left(\int d^4x\mathcal{L}_{QED}\right)} \tag{1.19}$$

with the classical action in the exponent being invariant under (1.16).

We want to understand the transformation properties of the measure, thus consider the general linear transformations

$$\psi(x) \mapsto \Delta(x)\psi \quad ; \quad \bar{\psi}(x) \mapsto \bar{\psi}(x)\Delta_c(x) \tag{1.20}$$

which generate a Jacobian factor for the measure

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \mapsto (J_cJ)^{-1}\mathcal{D}\bar{\psi}\mathcal{D}\psi. \tag{1.21}$$

The Jacobians $J = \det(\Delta)$, $J_c = \det(\Delta_c)$ appear with a negative power in the measure transformation because the fermionic fields ψ are treated as Grassmann variables in the path integral. To make sense of J, we write it as

$$J = \det(\Delta) = e^{\mathrm{tr}(\ln(\Delta))} = e^{\int d^4x\langle x|\mathrm{Tr}(\ln(\Delta))|x\rangle} \tag{1.22}$$

where tr is the trace of over the eigenvalues of Δ, and Tr is the Dirac trace. Written this way, it is easy to see that the transformation of the form $\Delta = e^{i\alpha}$ cancels out in the fermionic measure transformation (1.21), since the phases cancel out in the $(J_cJ)^{-1}$. However, for the transformation $\Delta = e^{i\beta\gamma^5}$, the phases add up in the Jacobian factor, and thus the fermionic measure part of the path integral under the axial symmetry transforms as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \mapsto \mathcal{D}\bar{\psi}\mathcal{D}\psi \cdot \exp\left(i\left(\int d^4\beta(x)\left(-2\mathrm{Tr}[\gamma^5\delta^4(x-x)]\right)\right)\right) \tag{1.23}$$

where we used the Dirac delta $\langle x|y\rangle = \delta^4(x-y)$. For convenience, we denote the trace part in the exponential as $\mathcal{A}(x) = -2\mathrm{Tr}[\gamma^5\delta^4(x-y)]$, and call it the anomaly function. Utilising the anomaly function, the full path integral now transforms as

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}Ae^{i\left(\int d^4x\mathcal{L}_{QED}\right)} \mapsto \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\cdot\exp\left(i\left(\int d^4x\mathcal{L}_{QED} - {J_A}^{\mu}\partial_\mu\beta + \beta(x)\mathcal{A}(x)\right)\right) \tag{1.24}$$

and so the infinitesimal path integral transformation only for the fermionic fields becomes

$$\delta \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\left(\int d^4x \mathcal{L}_{QED}\right)} = i \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \beta(x)[\partial_\mu {J_A}^\mu + \mathcal{A}(x)] e^{i\int d^4x \mathcal{L}_{QED}}. \tag{1.25}$$

Note that this is a mere change of variables and for some arbitrary $\beta(x)$, the infinitesimal fermionic measure transformation will not affect the path integral. Therefore, instead of the classical Nooether's conservation law for ${J_A}^\mu$, we have

$$\langle \partial_\mu {J_A}^\mu(x) \rangle = -\mathcal{A}(x). \tag{1.26}$$

It now remains to compute the anomaly function. At first sight, it seems like one cannot get a definite result for the anomaly function. Even though $\mathrm{Tr}[\gamma^5] = 0$, the delta function $\delta^4(x-x)$ is not well-defined, and so we have to introduce a regulator in the anomaly, to make the delta function meaningful. We can do this in a gauge-invariant manner, by introducing a differential regulator of the form $f\left(-\frac{\not{D}^2}{\Lambda^2}\right)$ so that it acts on the delta function before setting its argument to zero. The regulator f is a well-behaved function such as a Gaussian. We normalise it to $f(0) = 1$ and impose that $f(s)$ and its derivatives $f'(s)$ vanish at infinity. D appearing as the regulator is a covariant derivative, and Λ is some UV cut-off scale which we will send to infinity when evaluating the anomaly function. Note that we want to preserve the gauge invariance, so we take the argument in the regulator function of the form $\hat{\not{X}} = \gamma^\mu \hat{X}_\mu$ with $\hat{X}$ being some operator. Moreover, we do not take the argument in the regulator of the form $D_\mu D^\mu$ since we want to regulate both the Jacobian factor, and the fermion propagator $\not{D}^{-1}$. We will work in Euclidean space, so we introduce a Wick rotation in the spacetime coordinates $x_4 = ix_0$, and correspondingly $\gamma_4 = i\gamma_0$, $A_4 = iA_0$. Written this way, the differential operator is $i\not{D} = (i\partial_\mu + eA_\mu)\gamma_\mu$, with $\mu = 1, \ldots, 4$.

Now, expressing the delta function in the Fourier representation, we can evaluate the anomaly

$$\begin{aligned} \mathcal{A} &= -2 \lim_{\Lambda\to\infty} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 f\left(-\frac{\not{D}^2}{\Lambda^2}\right) e^{ik(x-y)}\right]_{y=x} \\ &= -2 \lim_{\Lambda\to\infty} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 f\left(-\frac{-(i\not{k} + \not{D})^2}{\Lambda^2}\right)\right] \end{aligned}. \tag{1.27}$$

Where in the second line, a derivative acting on the extreme right x gives zero, but not when acting on A_μ. We can rescale the momentum by the cut-off value as $k \to k/\Lambda$, and thus

$$\mathcal{A} = -2 \lim_{\Lambda\to\infty} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^5 f\left(-\left(i\not{k} + \frac{\not{D}}{\Lambda}\right)^2\right)\right]. \tag{1.28}$$

Now, the argument of the regulator may be written as

$$\left(i\not{k} + \frac{\not{D}}{\Lambda}\right)^2 = k^2 - \frac{k\cdot D}{\Lambda} - \left(\frac{\not{D}}{\Lambda}\right)^2. \tag{1.29}$$

Let us calculate what $\not{D}^2$ is

$$\begin{aligned}\not{D}^2 &= \frac{1}{4}\{D^\mu, D^\nu\}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{4}[D^\mu, D^\nu][\gamma^\mu, \gamma^\nu] \\ &= {D_\mu}^2 - \frac{ie}{4}F^{\mu\nu}[\gamma^\mu, \gamma^\nu]\end{aligned}. \tag{1.30}$$

To compute the anomaly, one can Taylor expand the regulator function in (1.28). When expanded, the anomaly function receives contributions only with factors $1/\Lambda^4$. Moreover, any odd power of $\not{D}$ gives an odd number of gamma matrices in the trace, and the trace property $\mathrm{Tr}[\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_n}] = 0$ can be used when n is an odd number. Putting all this together, we are left with

$$\mathcal{A} = -\int \frac{d^4k}{(2\pi)^4} f''(k^2)\mathrm{Tr}[\gamma^5\not{D}^4]. \tag{1.31}$$

Since we are working in Euclidean coordinates, the momentum integration part may be written as

$$\int d^4k f''(k^2) = \int_0^\infty 2\pi^2 k^3 dk f''(k^2). \tag{1.32}$$

We also said that f is a well-behaved function that vanishes evaluated at infinity (the f' vanishes as well). Thus, integrating by parts, one finds

$$\int d^4k f''(k^2) = \pi^2 \int_0^\infty s ds f''(s) = -\pi^2 \int_0^\infty ds f'(s) = \pi^2. \tag{1.33}$$

Using (1.30), the only non-vanishing contribution to the trace $\mathrm{Tr}[\gamma^5\not{D}^4]$ involves a product of four gamma matrices. Therefore, we can calculate the trace contribution to (1.31) as

$$\mathrm{Tr}[\gamma^5\not{D}^4] = \frac{e^2}{16}F^{\mu\nu}F^{\alpha\beta}\mathrm{Tr}\{\gamma_5[\gamma_\mu, \gamma_\nu][\gamma_\alpha, \gamma_\beta]\} \tag{1.34}$$

but working in Euclidean space, the trace of gamma matrices product gives an anti-symmetric tensor $\mathrm{Tr}\{\gamma_5[\gamma_\mu, \gamma_\nu][\gamma_\alpha, \gamma_\beta]\} = 16\epsilon_{\mu\nu\alpha\beta}$. Using results from (1.32) - (1.34) in equation (1.31) finally gives the anomaly function

$$\mathcal{A}(x) = -\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x). \tag{1.35}$$

Therefore, from the symmetry transformations we considered in the path integral (1.23) - (1.25), we can write the ABJ anomaly for an Abelian theory as

$$\langle\partial_\mu {J_A}^\mu(x)\rangle = \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x). \tag{1.36}$$

There are a few comments we could make about this anomaly. First of all, if we have a theory for N_f massless Dirac fermions, the anomaly equation can be written as

$$\langle\partial_\mu {J_A}^\mu(x)\rangle = \frac{e^2 N_f}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x). \tag{1.37}$$

Moreover, if we consider a theory for a massive Dirac fermion, then the divergence of the axial current becomes

$$\langle\partial_\mu {J_A}^\mu(x)\rangle = -2im\bar{\psi}\gamma^5\psi + \frac{e^2}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x) \tag{1.38}$$

and although the first term on the right-hand side of (1.38) explicitly breaks the axial symmetry, the anomaly contribution is still manifest in this equation. It is simple to generalise the anomaly derivation arguments discussed in above to a non-Abelian gauge theory. If a Dirac fermion sits in some representation R of a non-Abelian gauge group, then the right-hand side of (1.37) gets a trace factor over colour indices, and the compact version of the non-Abelian gauge anomaly becomes

$$\langle\partial_\mu {J_A}^\mu(x)\rangle = \frac{e^2}{8\pi^2}\mathrm{tr}_R(F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)). \tag{1.39}$$

where we have used the dual tensor form $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

1.4 Chiral Symmetry Breaking

Because of the ABJ anomaly (1.39), the global symmetry of $\mathcal{L}_{\text{QCD}}$ in the massless limit reduces to

$$G_F = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_V = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B. \tag{1.40}$$

This is known as a *chiral symmetry* [9] and both non-Abelian symmetry transformations still act as (1.9) and (1.10) in the flavour space, but the L and R are now the elements of $SU(N_f)$ instead of $U(N_f)$.

It should be noted that the *chiral symmetry* is only approximate because we pretend the quarks to be massless. This comes from the consideration of the QCD dynamical scale Λ_{QCD}. This is the scale at which the physics of QCD happens and if we have any quark with mass $m \ll \Lambda_{\text{QCD}}$ it turns out that it is appropriate to investigate the dynamics of QCD gauge fields in the massless limit. Moreover, in the presence of the masses, it is appropriate to understand the changes in physics in terms of corrections of order m/Λ_{QCD} [16]. The scale of QCD is $\Lambda_{\text{QCD}} \approx 200$ MeV and the only quarks that are below this scale are *up*, *down*, and *strange*

$$m_u = 2 \text{ MeV}, \quad m_d = 5 \text{ MeV}, \quad m_s = 93 \text{ MeV}. \tag{1.41}$$

While *up* and *down* quarks clearly satisfy the approximation $m_u, m_d \ll \Lambda_{\text{QCD}}$ and we can treat them as massless, the *strange* quark only satisfies $m_s < \Lambda_{\text{QCD}}$ and one may argue that the *strange* quark is not entirely massless. Nonetheless, it is worth seeing what happens to the physics when one pretends that there are 2 massless quarks and it is useful to see how this changes pretending when there are 3 massless quarks.

As we have seen in (1.8), the mass matrix, M, couples left-handed and right-handed fermions: $\bar{q}_{L_i} M q_{R_i}$. Moreover, the mass matrix does not commute with L and R transformations. If all quark masses were the same, e.g. $M = m \cdot 1_{4\times4}$, we would have [16]

$$\bar{q}_{L_i} M q_{R_i} \mapsto \bar{q}_{L_i} L^\dagger M R q_{R_i} = \bar{q}_{L_i} L^\dagger R M q_{R_i}. \tag{1.42}$$

The mass term is invariant only under simultaneous transformations $L = R$, so that $L^\dagger R = \mathbf{1}$. Hence, the chiral symmetry G_F is explicitly broken to

$$G_F = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B \mapsto SU(N_f)_{L=R} \otimes U(1)_B \tag{1.43}$$

Like in Section 1.2, it is important to think about *up*, *down* and *strange* quark masses to be small. Therefore, in reality, the chiral symmetry G_F is almost unbroken explicitly.

There is however a way to detect spontaneous chiral symmetry breaking by investigating the chiral order parameter (also known as the quark condensate)

$$\langle \bar{q}q \rangle = \langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle. \tag{1.44}$$

The chiral order parameter is only invariant under the simultaneous $L = R$ transformations but not under general chiral rotations. If the chiral symmetry were intact, the order parameter would vanish. However, because of the Goldstone's Theorem, for SSB to occur, $\langle \bar{q}q \rangle \neq 0$, and this only happens when the chiral symmetry is spontaneously broken [17].

1.5 Quark Condensate

There are two important phenomena that emerge in QCD-like theories: confinement and formation of quark condensate. The former phenomenon means that we do not observe individual quarks in the particle spectrum and instead, we see quarks coupled to colour singlets, including mesons (even quark number) and baryons (odd quark number). The latter phenomenon occurs due to the strong coupling dynamics of non-Abelian gauge theories such as QCD. It turns out that QCD-like theories give rise to a VEV of a similar form to (1.44)

$$\langle \bar{q}_{L_i} q_{R_j} \rangle = -\sigma \delta_{ij} \tag{1.45}$$

where σ has a dimension of $[\text{Mass}]^3$ since a fermion in $d = 3 + 1$ dimensions has a mass dimension $[q] = \frac{3}{2}$. The only relevant dimensional parameter in our theory is the strong coupling scale Λ_{QCD}, thus we expect that parametrically $\sigma \sim \Lambda_{\text{QCD}}{}^3$. One of the important immediate consequences of the quark condensate in (1.45) is that it breaks the chiral symmetry. To see this, let us consider how (1.45) transforms under general G_F transformations

$$\langle \bar{q}_{L_i} q_{R_j} \rangle \mapsto -\sigma \left(L^\dagger R \right)_{ij}. \tag{1.46}$$

We see that the condensate remains invariant only if $L = R$, and so we get the following symmetry breaking pattern

$$G_F = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B \mapsto SU(N_f)_V \otimes U(1)_B \tag{1.47}$$

where $SU(N_f)_V$ is the diagonal subgroup of $SU(N_f)_L \otimes SU(N_f)_R$. Spontaneous symmetry breaking occurs whenever the theory is invariant under some symmetry group, but the ground state is not. In the quark condensate context this means that the massless QCD theory exhibits a dynamical spontaneous symmetry breaking which is known as chiral symmetry breaking.

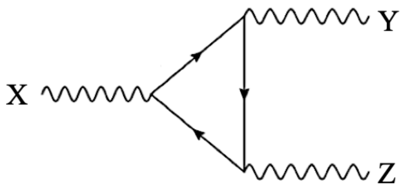


Fig. 1.1 Triangle diagram for fermions in the anomaly contribution. Indices X, Y, and Z indicate the symmetry groups.

We can now invoke the Goldstone's Theorem to find the vacuum manifold

$$\mathcal{M}_0 = \frac{SU(N_f)_L \otimes SU(N_f)_R}{SU(N_f)_V} \tag{1.48}$$

and thus the number of massless Goldstone bosons is given by

$$dim(\mathcal{M}_0) = {N_f}^2 - 1. \tag{1.49}$$

Consequently, if we pretend in our theory that there are $N_f = 2$ massless quarks (*up* and *down*), we should find 3 massless Goldstone bosons in our world which are known as *pions*. One could go further with an addition of an extra *strange* quark in the $N_f = 3$ massless regime and should find 8 massless Goldstone bosons which are called *kaons* and *eta*.

There is still a lack of complete understanding of how the quark condensate forms. It was first realised from experimental considerations since it explains the light meson spectrum, yet on theoretical grounds, the most compelling argument comes from numerical simulations of lattice theories. Nonetheless, there is an indirect argument that tells that the quark condensate forms whenever the theory confines. To explain this, first, we need to investigate how the anomaly contributions arise in QFT. In QFT, an anomaly is whenever a conserved symmetry of the classical field Lagrangian does not remain intact in the quantum path integral (and thus the quantum system). The anomaly contributions come from the measure of fermionic fields in the path integral and are proportional to the diagram shown in Figure 1.1. It will be useful to note that the anomaly contribution $\mathcal{A}$ in Figure 1.1. depends on the number of vertices associated with non-Abelian symmetries:

- For 0 non-Abelian vertices, $\mathcal{A}$ is proportional to the sum of cubes of charges q_i of fermions in the Abelian group.

- For 1 non-Abelian vertex, e.g. $X = SU(N)$ and $Y, Z = U(1)$, $\mathcal{A}$ is proportional to $\mathrm{Tr}T_R$ where T_R is the generator of $SU(N)$. These contributions usually vanish since T_R are usually traceless.

- For 2 non-Abelian vertices, e.g. $X = U(1)$ and $Y, Z = SU(N)$, $\mathcal{A}$ is proportional to $\sum_i {q^2}_i \mathrm{Tr}\left(\{{T^Y}_R, {T^Z}_R\}\right)$ where R denotes the representation in which fermions transform under $SU(N)$.

- For all 3 non-Abelian vertices, $\mathcal{A}$ is proportional to $\mathrm{Tr}\left({T^X}_R\{{T^Y}_R, {T^Z}_R\}\right)$.

Now we exploit the 't Hooft anomaly matching argument that says [12]

Consider a QFT in some UV regime with a global symmetry G_F that has some anomaly contribution $\mathcal{A}_{UV}$, then after deforming this theory under a Renormalization Group flow to some IR regime that produces anomaly contribution $\mathcal{A}_{IR}$ we have the following result:

Either anomalies match $\mathcal{A}_{UV} = \mathcal{A}_{IR}$ or the symmetry G_F is spontaneously broken.

Let us now recall the global symmetry of our theory in (1.40) and calculate the 't Hooft anomalies in the UV regime for the left-handed quarks. In general, let us consider an $SU(N_c)$ gauge theory, coupled to N_f massless fermions in fundamental representation. There is no anomaly in Figure 1.1 when $X = Y = Z = U(1)_V$ since this is a vector-like symmetry. However, there is an anomaly when $X = Y = Z = SU(N_f)_L$ and in fundamental representation it is proportional to $\mathcal{A} = N_c$. Each quark q_L transforms in a fundamental representation of $SU(N_f)_L$ and the vertex contribution count turns out to be 1, but each quark carries a colour index and after summation, it results in N_c. There is another anomaly contribution $\mathcal{A}'$ when $X = U(1)_V$ and $Y = Z = SU(N_f)_L$, and summing over the vertex contributions one finds $\mathcal{A}' = N_c$.

Now suppose that in the IR regime, G_F is unbroken. Following the 't Hooft anomaly matching argument, there must be massless fermions that should reproduce $\mathcal{A}$ and $\mathcal{A}'$. Moreover, in the context of QCD, these massless fermions should be bound states of quarks, either baryons or mesons. Mesons cannot reproduce these anomalies because they are bosons and we are left to consider baryons. Baryons contain N_c quarks and when N_c is even they also fail to reproduce 't Hooft anomalies which means that in the confining regime, chiral symmetry must be broken. When N_c is odd for baryons, one has to consider more complicated combinations of representations of G_F in which massless baryons sit, and it turns out that there are no possible combinations of massless baryons to reproduce $\mathcal{A}$ and $\mathcal{A}'$. This means that whenever OCD confines into weakly interacting colour singlets, the chiral symmetry must be broken.

1.6 Chiral Lagrangian

As we have seen in Section 1.5, chiral symmetry breaking χSB implies the existence of massless Goldstone bosons, and in general, for any broken continuous symmetry there exists a manifold of ground states $\mathcal{M}_0$. For the χSB (1.47), the different points on vacuum manifold (1.48) can be parameterised by the chiral condensate, which in general takes the form

$$\langle \bar{q}_{L_i} q_{R_j} \rangle = -\sigma U_{ij} \tag{1.50}$$

where $U = L^\dagger R \in SU(N_f)$. The Goldstone boson fields are not constant but rather are long-wavelength ripples of the condensate and thus we can assign the spacetime dependence $U = U(x)$. Since there are ${N_f}^2 - 1$ Goldstone bosons, one for each broken generator of the coset group (1.48), the effective low-energy theory then can be described in terms of the pion fields $\pi(x)$ [13]

$$U(x) = e^{\left(\frac{2i}{f_\pi}\pi(x)\right)} \tag{1.51}$$

where $\pi(x)$ is valued in the Lie algebra $su(N_f)$: $\pi(x) = \pi^a(x)T^a$. Here T^a are the matrix generators of $su(N_f)$ with $a = 1, \ldots, {N_f}^2 - 1$. Since pion scalar fields are conventionally assumed to have dimensions of mass $[\pi] = 1$, the f_π constant is some suitable energy scale with dimensions $[f_\pi] = 1$, and in general we expect $f_\pi \sim \Lambda_{\text{QCD}}$.

To construct the effective low energy theory for Goldstone modes $U(x)$, we require this theory to be invariant under the full chiral symmetry G_F (1.40) under which the transformations act on pion fields as

$$U(x) \mapsto L^\dagger U(x) R. \tag{1.52}$$

The construction of the effective chiral Lagrangian becomes straightforward because it is dictated by the symmetry transformations (1.52). The obvious term that will not appear in this theory is $\text{tr}(U^\dagger U)$ (trace over flavour indices a) since $U \in SU(N_f)$ and so $U^\dagger U = 1$. This is consistent with the fact that U Goldstone fields are massless, and we need to look for the terms that depend on spacetime derivatives of U. All the terms in the chiral Lagrangian have to be consistent with both $SU(N_f)_L \otimes SU(N_f)_R$ and Lorentz symmetries, and so there would be no terms with just a single derivative

$\partial_\mu U$, because its uncontracted Lorentz index would violate Lorentz invariance. The Lorentz invariance requirement applies to any higher - order derivative terms of U, and thus in general, we expect any term included in the chiral Lagrangian to have an even number of derivatives [14]. Next, we can exclude candidate terms like $\left(\mathrm{tr}\left(U^\dagger\partial_\mu U\right)\right)^2$ since $U^\dagger\partial_\mu U$ is a generator of $su(N_f)$ and thus it is traceless. We are left with the two candidates for the chiral Lagrangian

$$\mathrm{tr}\left(\partial^\mu U^\dagger\partial_\mu U\right), \quad \mathrm{tr}\left(U^\dagger\partial_\mu U\right)^2. \tag{1.53}$$

However, we use the fact that $U^\dagger U = 1$, and differentiating this product gives us the relation $U^\dagger\partial_\mu U = -\left(\partial_\mu U^\dagger\right)U$. We can use this relation to rewrite the second term in (1.53) in terms of the first term, which leaves us to the lowest order with the unique chiral Lagrangian

$$\mathcal{L}_\chi = \frac{{f_\pi}^2}{4}\mathrm{tr}\left(\partial^\mu U^\dagger\partial_\mu U\right). \tag{1.54}$$

We check that it is invariant under the global $SU(N_f)_L \otimes SU(N_f)_R$ transformations. First, we consider how transformations act on each term

$$\begin{aligned} U &\mapsto L^\dagger U R \\ \partial_\mu U \mapsto \partial_\mu\left(L^\dagger U R\right) = \partial_\mu L^\dagger U R + L^\dagger\partial_\mu U R + L^\dagger U\partial_\mu R &= L^\dagger\partial_\mu U R \\ U^\dagger &\mapsto R^\dagger U^\dagger L \\ \partial_\mu U^\dagger &\mapsto R^\dagger\partial_\mu U^\dagger L \end{aligned}. \tag{1.55}$$

Now we can apply these transformations to (1.54) and find the invariance

$$\mathcal{L}_\chi \mapsto \frac{{f_\pi}^2}{4}\mathrm{tr}\left(R^\dagger\partial^\mu U^\dagger L L^\dagger\partial_\mu U R\right) = \frac{{f_\pi}^2}{4}\mathrm{tr}\left(R R^\dagger\partial^\mu U^\dagger\partial_\mu U\right) = \mathcal{L}_\chi \tag{1.56}$$

since $L^\dagger L = R^\dagger R = 1$ and in the last step R was moved using the cyclicity property of the trace.

The chiral Lagrangian is non-renormalizable, which means that an infinite number of counter-terms need to be considered to regulate divergences in the interactions. However, this theory is restricted to low energies, which means that it makes sense up to some suitable energy scale f_π. If one considers one-loop contributions to pion interaction processes to the lowest order chiral Lagrangian $\mathcal{O}(p^4)$, it turns out that they scale as $p^4\ln(p^2)$ [14]. These so-called chiral logarithms cannot be generated from tree-level diagrams of higher-order terms such as $\mathcal{L}_4$. In fact, at low energies, the $\mathcal{O}(p^4)$ order contributions from $\mathcal{L}_4$ are dominated by one-loop contributions from $\mathcal{L}_2$.

1.7 Currents in Chiral Lagrangian

In Section 1.1 we started with a microscopic massless theory (1.8) and we ended up with a low energy theory (1.54) in Section 1.6. It would be useful to know how in general operators are mapped between high-energy and low-energy regimes. Since the physics of both theories in UV and IR regimes are constrained by symmetries, one could consider a particular class of operators: currents associated with the chiral symmetry.

Let us start with the construction of currents in the microscopic massless theory (1.8). Neglecting the quark masses in (1.8), we invoke Noether's theorem for the chiral $SU(N_f)_L \otimes SU(N_f)_R$ symmetry and find currents for each Weyl spinor

$$\begin{aligned} J^a{}_{L\mu}(x) &= \bar{q}_L(x)\gamma_\mu T^a q_L(x) \\ J^a{}_{R\mu}(x) &= \bar{q}_R(x)\gamma_\mu T^a q_R(x) \end{aligned} \tag{1.57}$$

where T^a are $su(N_f)$ generators. From the left-handed and right-handed currents, we now construct *vector* and *axial* currents

$$\begin{aligned} J^a{}_{\mu V}(x) &= J^a{}_{L\mu}(x) + J^a{}_{R\mu}(x) \\ = \bar{q}(x)\frac{1}{2}(1+\gamma_5)\gamma_\mu T^a \frac{1}{2}(1-\gamma_5) q(x) &+ \bar{q}(x)\frac{1}{2}(1-\gamma_5)\gamma_\mu T^a \frac{1}{2}(1+\gamma_5) q(x) \\ = \bar{q}(x)\frac{1}{2}(1+\gamma_5+1-\gamma_5)\gamma_\mu T^a q(x) &= \bar{q}(x)\gamma_\mu T^a q(x) \end{aligned} \tag{1.58}$$

$$\begin{aligned} J^a{}_{\mu A}(x) &= J^a{}_{L\mu}(x) - J^a{}_{R\mu}(x) \\ = \bar{q}(x)\frac{1}{2}(1+\gamma_5)\gamma_\mu T^a \frac{1}{2}(1-\gamma_5) q(x) &- \bar{q}(x)\frac{1}{2}(1-\gamma_5)\gamma_\mu T^a \frac{1}{2}(1+\gamma_5) q(x) \\ = \bar{q}(x)\frac{1}{2}(1+\gamma_5-1+\gamma_5)\gamma_\mu T^a q(x) &= \bar{q}(x)\gamma_5\gamma_\mu T^a q(x) \end{aligned} \tag{1.59}$$

where we used the anticommutation relation between γ_μ and γ_5.

We can now look for analogous currents in the low energy theory (1.54). Let us consider $SU(N_f)_L$ first and parameterize transformations

$$L = \exp\left(i\alpha^a{}_L T^a\right) \approx 1 + i\alpha^a{}_L T^a + \dots \tag{1.60}$$

and promote ${\alpha^a}_L = {\alpha^a}_L(x)$. Under this, we find the infinitesimal transformations to first order in ${\alpha^a}_L$

$$\begin{aligned} U &\mapsto L^\dagger U = \left(1 - i{\alpha^a}_L T^a\right) U \\ U^\dagger &\mapsto U^\dagger \left(1 + i{\alpha^a}_L T^a\right) \\ \partial_\mu U &\mapsto \left(1 - i{\alpha^a}_L T^a\right) \partial_\mu U - U i \partial_\mu {\alpha^a}_L T^a \\ \partial_\mu U^\dagger &\mapsto \partial_\mu U^\dagger \left(1 + i{\alpha^a}_L T^a\right) + i \partial_\mu {\alpha^a}_L T^a U^\dagger \end{aligned} \tag{1.61}$$

and use these to obtain the infinitesimal transformation for $\mathcal{L}_\chi$

$$\begin{aligned} \delta\mathcal{L}_\chi &= \frac{{f_\pi}^2}{4} \mathrm{tr}\left(iU\partial_\mu {\alpha^a}_L T^a \partial^\mu U^\dagger - i\partial_\mu U \partial^\mu {\alpha^a}_L T^a U^\dagger\right) \\ &= \frac{{f_\pi}^2}{4} i \left(\partial^\mu {\alpha^a}_L\right) \mathrm{tr}\left(T^a[\partial_\mu U^\dagger U - U^\dagger \partial_\mu U]\right) \\ &= \frac{{f_\pi}^2}{2} i \left(\partial^\mu {\alpha^a}_L\right) \mathrm{tr}\left(T^a \partial_\mu U^\dagger U\right) \end{aligned} \tag{1.62}$$

where in the last step a relation $U^\dagger \partial_\mu U = -\left(\partial_\mu U^\dagger\right) U$ was used. We thus obtain the left-handed current as

$$ {J^a}_{\mu L} = \frac{\partial \delta\mathcal{L}_\chi}{\partial\left(\partial^\mu {\alpha^a}_L\right)} = i\frac{{f_\pi}^2}{2} \mathrm{tr}\left(T^a \partial_\mu U^\dagger U\right). \tag{1.63}$$

Following the completely analogous computations for $SU(N_f)_R$, one finds the right-handed current

$$ {J^a}_{\mu R} = -i\frac{{f_\pi}^2}{2} \mathrm{tr}\left(T^a U \partial_\mu U^\dagger\right) \tag{1.64}$$

and the analogous vector and axial currents are found to be

$$\begin{aligned} {J^a}_{\mu V} = {J^a}_{\mu L} + {J^a}_{\mu R} = i\frac{{f_\pi}^2}{2} \mathrm{tr}\left(T^a[\partial_\mu U^\dagger, U]\right) \\ {J^a}_{\mu A} = {J^a}_{\mu L} - {J^a}_{\mu R} = -i\frac{{f_\pi}^2}{2} \mathrm{tr}\left(T^a\{\partial_\mu U^\dagger, U\}\right) \end{aligned}. \tag{1.65}$$

We can expand both currents (1.63) and (1.64) to the leading order in terms of the pion fields, e.g.

$$ {J^a}_{\mu L} \approx i\frac{{f_\pi}^2}{2} \mathrm{tr}\left(T^a\left(-i\frac{2T^b\partial_\mu \pi^b(x)}{f_\pi} + \ldots, 1 + i\frac{2T^c\pi^c(x)}{2} + \ldots\right)\right) = -\frac{f_\pi}{2}\partial_\mu \pi^a(x) \tag{1.66}$$

and similarly

$$J^a{}_{\mu R} \approx \frac{f_\pi}{2} \partial_\mu \pi^a(x). \tag{1.67}$$

Using equations (1.65) - (1.67), one finds that there is a non-vanishing matrix element between the vacuum state $|0\rangle$ and one Goldstone boson state $|\pi^b(p)\rangle$ for the axial current

$$\langle 0|J^a{}_{\mu A}|\pi^b(p)\rangle = \langle 0|f_\pi \partial_\mu \pi^a(x)|\pi^b(p)\rangle = -f_\pi \delta^{ab} \partial_\mu e^{(-ip\cdot x)} = i f_\pi p_\mu \delta^{ab} e^{(-ip\cdot x)}. \tag{1.68}$$

One can simply check that although the $SU(N_f)_L \otimes SU(N_f)_R$ symmetry is broken, the diagonal subgroup $SU(N_f)_V$ survives, and the analogous matrix element for a vector current vanishes

$$\langle 0|J^a{}_{\mu V}|\pi^b(p)\rangle = \langle 0| - \frac{f_\pi}{2} \partial_\mu \pi^a(x) + \frac{f_\pi}{2} \partial_\mu \pi^a(x)|\pi^b(p)\rangle = 0. \tag{1.69}$$

This approach to chiral symmetry breaking is historically known as *current algebra* [15] and equation (1.68) means that for the broken $SU(N_f)_L \otimes SU(N_f)_R$, acting with current operators on vacuum states gives rise to the massless pion particles. It was used in calculations for interaction process amplitudes, including the charged pion decay $\pi^+ \mapsto \mu^+ + \nu_\mu$.

1.8 Charged pion decay

One of the examples of current operator application could be seen in the charged pion decay which is described by the annihilation between u and $\bar{d}$ quarks that form the W^+ boson propagator which decays to leptons μ^+ and ν_μ in the final state. At low energies, one can integrate-out the W - boson contributions in the path integral, and the decay can be described by Fermi's four-fermion interaction

$$\mathcal{L}_{4\mathrm{F}} = \frac{G_F}{\sqrt{2}} J^\mu J_\mu{}^\dagger \tag{1.70}$$

where G_F is the Fermi constant and $J^\mu = V_{ud} \bar{u} \gamma^\mu (1-\gamma_5)\, d + \bar{\nu}_\mu \gamma^\mu (1-\gamma_5)\, \mu + \ldots$ with V_{ud} being the CKM matrix element. The S-matrix element is then calculated

$$\langle \mu^+, \nu_\mu|\mathcal{L}_{4\mathrm{F}}|\pi^+\rangle = \frac{G_F}{\sqrt{2}} \int d^4x \langle \mu^+, \nu_\mu|J^\mu J_\mu{}^\dagger|\pi^+\rangle \tag{1.71}$$

and the matrix element can be factorised as

$$\langle \mu^+, \nu_\mu | J^\mu J_\mu{}^\dagger | \pi^+ \rangle = \langle \mu^+, \nu_\mu | \bar{\nu}_\mu \gamma^\mu \left(1 - \gamma_5\right) \mu | 0 \rangle \langle 0 | V_{ud} \bar{u} \gamma^\mu \left(1 - \gamma_5\right) d | \pi^+ \rangle. \tag{1.72}$$

The leptonic part $\langle \mu^+, \nu_\mu | \bar{\nu}_\mu \gamma^\mu \left(1 - \gamma_5\right) \mu | 0 \rangle$ can be calculated perturbatively, and the hadronic part $\langle 0 | V_{ud} \bar{u} \gamma^\mu \left(1 - \gamma_5\right) d | \pi^+ \rangle$ can now be related to (1.68) where we have to use $\pi^+ = \frac{1}{\sqrt{2}} \left(\pi^1 + i\pi^2\right)$. Using these relations, the matrix element for the pion decay can be written as

$$\mathcal{M} \left(\pi^+ \to \mu^+ + \nu_\mu\right) = \frac{G_F}{2} f_\pi p^\mu V_{ud} \bar{\nu}_\mu \gamma^\mu \left(1 - \gamma_5\right) \mu \tag{1.73}$$

where $\bar{\nu}_\mu$ and μ should be treated as spinors for ν_μ and μ^+ respectively. Lastly, the differential decay rate in the pion rest frame is given by

$$d\Gamma = \frac{|\mathcal{M}|^2}{2M_\pi} \frac{d^3 p_\nu}{2E_\nu (2\pi)^3} \frac{d^3 p_\mu}{2E_\mu (2\pi)^3} (2\pi)^4 \delta^4 (p_\pi - p_\mu - p_\nu). \tag{1.74}$$

Squaring the spinor part of the matrix element $\bar{\nu}_\mu p^\mu \gamma^\mu \left(1 - \gamma_5\right) \mu$ and summing over the spin projections, one gets a contribution of $4 m_\mu{}^2 \frac{M_\pi{}^2}{2} \left(1 - \frac{m_\mu{}^2}{m_\pi{}^2}\right)$ and (1.74) becomes

$$d\Gamma = \frac{G_F{}^2}{16\pi^2} |V_{ud}|^2 f_\pi{}^2 m_\mu{}^2 M_\pi \left(1 - \frac{m_\mu{}^2}{m_\pi{}^2}\right) \int \frac{d^3 p_\mu}{E_\mu E_\nu} \delta \left(M_\pi - E_\mu - E_\nu\right). \tag{1.75}$$

We can write $d^3 p_\mu = p_\mu{}^2 dp_\mu d\Omega_\mu$ and integrating (1.75) over the phase space yields the decay rate

$$\Gamma \left(\pi^+ \to \mu^+ + \nu_\mu\right) = \frac{G_F{}^2}{8\pi} f_\pi{}^2 |V_{ud}|^2 m_\mu{}^2 M_\pi \left(1 - \frac{m_\mu{}^2}{m_\pi{}^2}\right)^2. \tag{1.76}$$

Through this example one can make sense of the f_π quantity: it is the scale that determines the decay width of the pion and thus it is usually called the *pion decay constant* [14].

1.9 Neutral pion decay

By the 1960s, the interpretation of the pion as a massless Goldstone boson associated with a spontaneously broken $SU(N_f) \otimes SU(N_f)$ symmetry led to a number of fruitful results. As we have seen in the previous section, the chiral Lagrangian turned out to be successful in describing numerous pion scattering processes, such as the charged pion decay, in the current algebra formalism. Moreover, the chiral Lagrangian allows one to read off the *Gell-Mann - Oakes and Renner* relations for meson masses [15].

However, this interpretation of pion also had a few failures. Among these, the most prominent was the neutral pion decay mode $\pi^0 \to \gamma + \gamma$. The neutral pion decay was initially computed by Steitenberg [21] in the 1940s, assuming a coupling to nucleons $N = (p, n)$ of the form $G_{\pi N}\pi^a \bar{N}\gamma^5\sigma^a N$. This gives a close estimate to the observed value, but this calculation is unfortunately wrong. This indicated that something anomalous is going on in the π^0 decay. Similar problems arose when investigating other decay modes, such as $\eta^0 \to \gamma + \gamma$.

To understand what is going on, it would be useful for us to return to the discussion of the ABJ anomaly. Gauging a subgroup $U(1)_{EM} \subset SU(2)_V$ introduces an anomaly for axial currents.

We can import our calculation for the ABJ anomaly (1.36) for quarks coupled to the electromagnetic field:

$$\partial_\mu {J_A}^\mu = \frac{\epsilon_{\mu\nu\rho\sigma}}{16\pi^2} F^{\mu\nu} F^{\rho\sigma} \mathrm{tr}\{Q^2\tau^a\} \tag{1.77}$$

where the charge matrix $Q = \mathrm{diag}[\frac{2}{3}, -\frac{1}{3}]$, and the only non-vanishing generator τ^a in the trace is $\tau^3 = \frac{\sigma^3}{2}$, with σ^3 being the Pauli spin matrix. Putting everything together in (1.77), one calculates the divergence for the axial current

$$\partial_\mu {J_A}^\mu = \frac{e^2 N_c}{96\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{1.78}$$

where N_c is the number of quark colours.

Next, let us consider a local chiral transformation of the form $A = e^{i\frac{1}{2}\tau^3\beta}$, so that the chiral action transforms infinitesimally as

$$\delta S_\chi = \int d^4x \partial_\mu(\beta {J_A}^\mu) = \frac{e^2 N_c}{96\pi^2} \int d^4x \beta \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \tag{1.79}$$

The anomaly may then be incorporated in the chiral Lagrangian, modifying the action, so that it satisfies

$$\delta S_\chi = \frac{e^2 N_c}{96\pi^2} \int d^4x \beta \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{1.80}$$

when the chiral field transforms as

$$\delta U(\pi) = i\beta\{\frac{1}{2}\tau^3, U(\pi)\}. \tag{1.81}$$

Under this infinitesimal variation $\delta U(\pi)$, the pion field transforms as $\delta\pi^0 = \beta f_\pi$, so one can write the additional anomalous piece to the chiral Lagrangian as

$$\mathcal{L}_{anom}(x) = \frac{e^2 N_c}{96\pi^2 f_\pi}\pi^0(x)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x). \tag{1.82}$$

Now, for the decay mode $\pi^0 \to \gamma + \gamma$, the amplitude may be written as

$$\mathcal{M} = \langle\gamma(\mathbf{q}_1, \varepsilon_1)\gamma(\mathbf{q}_2, \varepsilon_2)|\mathcal{L}_{anom}(0)|\pi^0(\mathbf{p})\rangle \tag{1.83}$$

where $\varepsilon_1, \varepsilon_2$ are poralisations of the out-coming photons and the momentum relations

$$q_1{}^2 = q_2{}^2 = 0 \quad ; \quad p^2 = m_\pi{}^2 \quad ; \quad \mathbf{p} = \mathbf{q}_1 + \mathbf{q}_2. \tag{1.84}$$

Using the $\mathcal{L}_{anom}$ expression, one obtains

$$\mathcal{M} = 8\frac{e^2 N_c}{96\pi^2}\pi^0(x)\beta\epsilon_{\mu\nu\rho\sigma}\varepsilon_{1\mu}{}^* q_{1\nu}\varepsilon_{2\rho}{}^* q_{2\sigma}. \tag{1.85}$$

Thus the decay rate is

$$\Gamma(\pi^0 \to \gamma + \gamma) = \frac{1}{2}\frac{1}{2m_\pi}\sum_{\varepsilon_1,\varepsilon_2}\int \frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}|\mathcal{M}|^2 \tag{1.86}$$

with the factor 1/2 appearing in the front since the two photons are treated as identical. Summing over the photon spins, one gets $\sum_\varepsilon \varepsilon_\mu\varepsilon_\nu{}^* = -g_{\mu\nu}$, with $g_{\mu\nu}$ being metric. Contracting anti-symmetric tensors gives $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta} = -2(\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\beta{}^\rho\delta_\alpha{}^\sigma)$. Using these results, and the momentum relations (1.84), one gets the spinor contribution part squared in (1.86)

$$\sum_{\varepsilon_1,\varepsilon_2}|\epsilon^{\mu\nu\rho\sigma}\varepsilon_{1\mu}{}^* q_{1\nu}\varepsilon_{2\rho}{}^* q_{2\sigma}|^2 = 2(q_1 \cdot q_2)^2. \tag{1.87}$$

Since $2q_1 \cdot q_2 = m_\pi{}^2$, the total summation contribution of the matrix element becomes

$$\sum_{\varepsilon_1,\varepsilon_2}|\mathcal{M}|^2 = \frac{e^2 N_c{}^2 m_\pi{}^4}{16\pi^2 f_\pi{}^2}. \tag{1.88}$$

The phase space integration gives

$$\int \frac{d^3q_1}{|\mathbf{q}_1|}\frac{d^3q_2}{|\mathbf{q}_2|}\delta^4(\mathbf{p} - \mathbf{q}_1 - \mathbf{q}_2) = 2\pi \tag{1.89}$$

and so the predicted pion decay rate is

$$\Gamma(\pi^0 \to \gamma + \gamma) = \frac{e^2 {N_c}^2 {m_\pi}^3}{256\pi^3 {f_\pi}^2} = \left(\frac{N_c}{3}\right)^2 \times 1.11 \times 10^{16} s^{-1}. \tag{1.90}$$

This is in good agreement with the observed decay rate value when $N_c = 3$. The success of this calculation was one of the first indications that there are three colours of quarks in the Standard Model.

Chapter 2

Topological aspects of the *ABJ* anomaly

There is one more subtle aspect that follows from the derivation of the *ABJ* anomaly (1.37). Since this derivation only relied on the path integral measure and did not involve perturbation calculations, it seems that the anomaly equation is 1-loop exact. For deep topological reasons, this anomaly turns out to be indeed 1-loop exact. The 1-loop exactness of the anomaly was first shown by Adler and Bell using higher-order loop diagrammatic analysis [2]. Here we provide a more modern viewpoint on this.

2.1 The Atiyah-Singer index theorem

Consider again the differential Dirac operator in Euclidean space for a general gauge field A_μ: $i\not{D} = (i\partial_\mu + A_\mu)\gamma^\mu$. It is hermitian, thus it has orthonormal spinor eigenfunctions ϕ_k, with real eigenvalues λ_k

$$\begin{aligned} i\not{D}\phi_k &= \lambda_k \phi_k \\ \int d^4x {\phi_k}^\dagger(x)\phi_{k'}(x) &= \delta_{k,k'} \end{aligned} . \tag{2.1}$$

We also consider again the local chiral transformations on fields $\psi(x)$ and $\bar{\psi}(x)$ as

$$\begin{aligned} \delta\psi(x) &= i\alpha(x)t\gamma^5\psi(x) \\ \delta\bar{\psi}(x) &= -i\alpha(x)\bar{\psi}(x)t\gamma^5 \end{aligned} \tag{2.2}$$

where t is a Hermitian matrix. We assume that t commutes with the Dirac operator, so we can choose ϕ_k such that $t\phi_k = t_k\phi_k$. These eigenfunctions satisfy the completeness relation as

$$\sum_k {\phi_k}^\dagger(x)\phi_k(y) = \delta^4(x-y)\mathbf{1}_{4\times 4}. \tag{2.3}$$

So, the anomaly function (1.27) can be written now as

$$\begin{aligned} \mathcal{A}(x) &= -2\lim_{\Lambda\to\infty} \mathrm{Tr}\left[\gamma^5 t f\left(-\frac{\not{D}^2}{\Lambda^2}\right)\sum_k {\phi_k}^\dagger(x)\phi_k(x)\right] \\ &= -2\lim_{\Lambda\to\infty}\sum_k t_k f\left(\frac{{\lambda_k}^2}{\Lambda^2}\right)\left({\phi_k}^\dagger(x)\gamma^5\phi_k(x)\right) \end{aligned}. \tag{2.4}$$

Let us consider the case when $\lambda_k \neq 0$. This means that for every normalised eigenfunction ϕ_k with λ_k, there is another normalised eigenfunction ${\phi_k}^- = \gamma^5\phi_k$, such that ${\lambda_k}^- = -\lambda_k$. Both ϕ_k and ${\phi_k}^-$ are eigenvectors of a hermitian Dirac operator, and they have different eigenvalues. Therefore, we have $\int d^4x {\phi_k}^\dagger(x)\gamma^5\phi_{k'}(x) = 0$.

Now, we are left with the case for eigenfunctions with $\lambda_k = 0$. Since γ^5 anticommutes with $i\not{D}$, these eigenfunctions can be chosen as orthonormal eigenfunctions of $i\not{D}$: ϕ_u, ϕ_v with $\lambda = 0$ for $i\not{D}$, and $\lambda = \pm 1$ for γ^5

$$\begin{aligned} i\not{D}\phi_u = 0 \quad &; \quad \gamma^5\phi_u = \phi_u \\ i\not{D}\phi_v = 0 \quad &; \quad \gamma^5\phi_v = -\phi_v \end{aligned}. \tag{2.5}$$

Recall that we required the regulator $f(s)$ to be a well-defined function such that $f(0) = 1$, so using relations (2.5) in (2.4), one gets the sum for the anomaly

$$\mathcal{A}(x) = -2\left[\sum_u t_u\left({\phi_u}^\dagger(x)\phi_u(x)\right) - \sum_v t_v\left({\phi_v}^\dagger(x)\phi_v(x)\right)\right]. \tag{2.6}$$

Since the eigenfunctions ϕ_u, ϕ_v are normalized, integrating both sides of (2.6) in Euclidean space gives

$$\int d^4x_E\mathcal{A}(x) = -2\left[\sum_u t_u - \sum_v t_v\right]. \tag{2.7}$$

The sums over u and v are running over the left-handed and right-handed zero modes of the Dirac operator $i\not{D}$. So, we may re-express (2.7) as

$$\int d^4x_E\mathcal{A}(x) = -2[n_+ - n_-] \tag{2.8}$$

where the total number of zero modes for $i\not{D}$ is $n_+ + n_-$. Therefore, using this result for integrating both sides of (1.37) yields the relation

$$n_+ - n_- = -\frac{1}{32\pi^2}\int d^4x_E \epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x). \tag{2.9}$$

This is known as the *Atiyah-Singer index theorem* [3]. One of the things it shows is that under variations in the gauge fields on the right-hand side of (2.9), the integral cannot change smoothly, but rather by integers, and thus the the integral depends on the topology of the gauge fields.

2.2 The *Cartan-Maurer Integral Invariant*

To investigate the topology of the gauge field in the integral of (2.9), it will be useful for us to discuss the *Cartan-Maurer Integral Invariant.*

Let us consider a compact, n-dimensional manifold S with coordinates $x_1, x_2, \ldots, x_n$. Assuming we have a mapping $g : S \mapsto G$, with G being a manifold of non-singular matrices, we define a functional of $g(x)$, known as the $Cartan-Maurer form$:

$$I(g) = \int d^n x \epsilon^{i_1\ldots i_n}\mathrm{Tr}[g^{-1}\left(\partial_{i_1}g\right)\ldots g^{-1}\left(\partial_{i_n}g\right)] \tag{2.10}$$

with $\partial_i = \frac{\partial}{\partial x^i}$, and g^{-1} being the matrix inverse of $g : g^{-1}g = gg^{-1} = \mathbf{1}$. The symbol $\epsilon^{i_1\ldots i_n}$ is totally anti-symmetric with $\epsilon^{12\ldots n} = 1$. Therefore, we can use the fact that $\epsilon^{i_1\ldots i_n} = -(-1)^n\epsilon^{i_2\ldots i_n i_1}$ and the cyclicity property of the trace to see that $I(g)$ vanishes with an even number of dimensions, n.

A more useful property of $I(g)$ is that it is invariant under infinitesimal transformations $g \to g + \delta g$. Using the cyclicity of the trace, we get the transformation

$$\delta I(g) = n\int d^n x \epsilon^{i_1\ldots i_n}\mathrm{Tr}[g^{-1}\left(\partial_{i_1}g\right)\ldots\delta\left(g^{-1}\partial_{i_n}g\right)] \tag{2.11}$$

where a factor of n in front comes from the fact that the change for each term $g^{-1}\partial_{i_m}g$ gives the same contribution to the change in $I(g)$. The last factor in (2.11) can be written as

$$\delta\left(g^{-1}\partial_{i_n}g\right) = -g^{-1}\delta g g^{-1}\frac{\partial g}{\partial x^{i_n}} + g^{-1}\frac{\partial\delta g}{\partial x^{i_n}} = g^{-1}\partial_{i_n}\left(\delta g g^{-1}\right)g. \tag{2.12}$$

Now, we use (2.12) for the integration of $\delta I(g)$ by parts. The differentiation $\frac{\partial}{\partial x^{i_n}}$ gives no contribution when acting on partial derivatives $\frac{\partial g}{\partial x^{i_m}}$ since the $\epsilon^{i_1\ldots i_n}$ is anti-symmetric. When $\frac{\partial}{\partial x^{i_n}}$ acts on the remaining g^{-1} terms in (2.11) (there are $n-1$ of such), they give equal contributions with an alternating sign. Since we showed that for non-vanishing $I(g)$, n is odd, $n-1$ is an even number, and thus $\delta I(g) = 0$. Therefore, $I(g)$ is invariant under smooth variations in g, and thus only changes by an integer.

Now, suppose that we continuously deform the manifold G into an n-dimensional Lie group with parameters x for $g(x)$. This deformation, followed by a Lie group transformation with parameters ϕ, results in a Lie group transformation with parameters $y(x,\phi)$. This reads as $g(\phi)g(x) = g(y(x,\phi)$. Differentiating this relation with respect to y and and multiplying by the inverse of $g(y(x,\phi)$ one gets

$$\frac{\partial x^i}{\partial y^j} g^{-1}(x) \frac{\partial g(x)}{\partial x^i} = g^{-1}(y) \frac{\partial g(y)}{\partial y^j} \tag{2.13}$$

so that the *Cartan-Maurer Integral* becomes

$$\begin{aligned} I(g) &= \int d^n y \, \epsilon^{j_1 \dots j_n} \mathrm{Tr}\left[g^{-1}(y) \frac{\partial g(y)}{\partial y^{j_1}} \dots g^{-1}(y) \frac{\partial g(y)}{\partial y^{j_n}} \right] \\ &= \int d^n y \, \det\left(\frac{\partial x}{\partial y}\right) \epsilon^{i_1 \dots i_n} \mathrm{Tr}\left[g^{-1}(x) \frac{\partial g(x)}{\partial x^{i_1}} \dots g^{-1}(x) \frac{\partial g(x)}{\partial x^{i_n}} \right]. \end{aligned} \tag{2.14}$$

The $\det\left(\frac{\partial x}{\partial y}\right)$ can be related to the Lie group metrics $\gamma_{ij}(x)$ of x and y such that $\det\left(\frac{\partial x}{\partial y}\right) = \sqrt{\frac{\det(\gamma(y))}{\det(\gamma(x))}}$ where $\gamma_{ij}(x) = -\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g)$. The ϕ parameters of a Lie group transformation are arbitrary, and we consider x and y as independent variables. Thus, we may evaluate $I(g)$ at any value of x, e.g. $x^i = 0$, and normalise the coordinates x^i with generators t_i:

$$g(x) \to 1 + 2ix^i t_i, \quad as\ x \to 0. \tag{2.15}$$

With this normalisation, $I(g)$ in (2.14) becomes

$$I(g) = \frac{\epsilon^{i_1 \dots i_n}(2i)^n}{\sqrt{\det\gamma(0)}} \int d^n y \sqrt{\det\gamma(y)} \, \mathrm{Tr}[t_{i_1} \dots t_{i_n}]. \tag{2.16}$$

Working in 3-dimensional space, we are interested in continuous maps $\mathbf{S}^3 \mapsto G$ with elements $g(x) \in SU(2) \subset G$. This mapping is characterised by the third homotopy group $\pi_3(G) = Z$ [8]. A general $SU(2)$ element can be expressed as $g(x) = x_4 + 2i\mathbf{x}\cdot\tau$, with τ being the Pauli spin matrices, and $x_4, \mathbf{x} \in R$ with ${x_4}^2 + \mathbf{x}^2 = 1$. For this parametrisation, the metric calculation for x is straightforward and it reads as

$$\gamma_{ij}(x) = \delta_{ij} + \frac{x_i x_j}{1 - |\mathbf{x}|^2} \quad \Rightarrow \quad \det\gamma(x) = \frac{1}{1 - |\mathbf{x}|^2}. \tag{2.17}$$

Thus, the $I(g)$ in (2.16) becomes

$$I(g) = -8i\epsilon^{ijk} \int d^3x \, \frac{\mathrm{Tr}[t_i t_j t_k]}{\sqrt{1 - |\mathbf{x}|^2}}. \tag{2.18}$$

We may use the Pauli matrix identities $t_i t_j = \frac{1}{4}\delta_{ij} + \frac{1}{2}i\epsilon^{ijk}$ and $\mathrm{Tr}(t_i t_j) = \frac{1}{2}\delta_{ij}$, so that the trace part of the integral

$$-8i\epsilon^{ijk}\mathrm{Tr}[t_i t_j t_k] = -2(i)^2\epsilon^{ijk}\epsilon^{ijk} = 12. \tag{2.19}$$

Moreover, the integral part of (2.18) runs twice over the unit sphere, since x_4 can be either positive or negative (for 1 dimension we have we have a standard exponential identity mapping $\mathbf{S}^1 \mapsto U(1)$) and so we may compute it as

$$\int \frac{d^3x}{\sqrt{1 - |\mathbf{x}|^2}} = 2\int_0^1 \frac{4\pi r^2 dr}{\sqrt{1 - r^2}} = 2\pi^2. \tag{2.20}$$

Since $I(g)$ is topologically invariant under δg, it can be regarded as $I(c)$ for some topology class c to which g belongs. In our considered 3-dimensional case, integrals $I(c)$ furnish a representation of the homotopy group $\pi_3(G)$ in the sense that $I(c_1 \times c_2) = I(c_1) + I(c_2)$. Hence, for homotopy classes c , $c \times c$, etc, we have $I(c^n) = nI(c)$. For the class c mappings homotopic to the identity map, from (2.18) - (2.20) we have $I(c) = 24\pi^2$, and thus $I(c^n) = 24\pi^2 n$, with the integer n known as the winding number.

2.3 Relation to Instantons

Having discussed the topological features of the *Cartan-Maurer Integral Invariant*, let us investigate the right-hand side integral of (2.9) dependence on the topology of the gauge field configurations.

The starting point is to consider the action for a pure Yang-Mills theory in 4-dimensional Euclidean space

$$S[A] = \frac{1}{4}\int d^4x \, F_{\mu\nu}F^{\mu\nu}. \tag{2.21}$$

Let us consider the following gauge transformation

$$A_\mu \to U A_\mu U^{-1} + i U^{-1}\partial_\mu U \tag{2.22}$$

where $U(\hat{x})$ is a direction-dependent element of gauge group G, and it does not depend on the magnitude $\mathbf{x}$.

Let us partially fix the gauge, demanding $A_0(x) = 0$. Now, are there any gauge field configurations that would make the action (2.21) finite? It turns out that this can be achieved by requiring $F_{\mu\nu}(\mathbf{x})$ to vanish as $|\mathbf{x}| \to \infty$. This can be done for a field $A_\mu(\mathbf{x})$ that vanishes slowly as $\sim 1/|\mathbf{x}|$, as long as it approaches a pure gauge in the limit $|\mathbf{x}| \to \infty$

$$A_\mu(\mathbf{x}) \to U^{-1}\partial_\mu U. \tag{2.23}$$

Moreover, for the finite action, we also need $U(\hat{x}_1) \to 1$ as $\mathbf{x} \to \infty$, for any one direction $\hat{x}_1$. In this way, for any finite action in 3-dimensional space, field configurations involve a map $U(x) : \mathbf{S}^3 \mapsto G$, which is characterized by the third homotopy group $\pi_3(G) = Z$. Such 4-dimensional field configurations are known as instantons [5].

With the asymptotic behaviour of the gauge field A_μ in mind, we can write down the *Cartan-Maurer Integral Invariant* as

$$\begin{aligned} I(U) &= \int d^3y\, \epsilon^{abc}\mathrm{Tr}\left[U^{-1}\partial_a U U^{-1}\partial_b U U^{-1}\partial_c U\right] \\ &= -i \lim_{\mathbf{x}\to\infty} \mathbf{x}^3 \int d^3y\, \epsilon^{abc}\frac{\partial \hat{x}_\mu}{\partial y^a}\frac{\partial \hat{x}_\nu}{\partial y^b}\frac{\partial \hat{x}_\rho}{\partial y^c}\mathrm{Tr}[A_\mu A_\nu A_\rho] \end{aligned} \tag{2.24}$$

where y^a with $a = 1, 2, 3$ are parameters which specify the direction of the 4-vector $\hat{x}$. In Euclidean space, term $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ can be expressed as the divergence of the Chern-Simons current K^μ:

$$\begin{aligned} \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= \partial_\mu K^\mu\ , \\ K_\mu &= \epsilon^{\mu\nu\rho\sigma}\left(A_\nu{}^i F^i_{\rho\sigma} - \frac{1}{3}f_{ijk}A_\nu{}^i A_\rho{}^j A_\sigma{}^c\right) \end{aligned} \tag{2.25}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is totally anti-symmetric ($\epsilon_{1234} = 1$), and f_{ijk} are the structure constants of the gauge group, so

$$\mathrm{Tr}[t_i t_j] = \frac{1}{2}N\delta_{ij} \tag{2.26}$$

with N being a constant which depends on the representation of the trace. In the limit $|\mathbf{x}| \to \infty$, we argued that we want $F_{\mu\nu}$ to vanish, and so Chern-Simons current becomes

$$K_\mu = \frac{4i}{3N}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[A_\mu A_\nu A_\rho] \tag{2.27}$$

and thus we may write the integral (2.24) as

$$I(U) = -\frac{3N}{8}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{2.28}$$

For our instanton field configurations, with non-vanishing Euclidean integral, we want to show that they minimise the action (2.21). To do this, consider the Bogomol'nyi inequality:

$$\begin{aligned} \int d^4x\ \left(F_{\mu\nu} \pm \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right)^2 \geq 0 \\ \int d^4x\ \left(F_{\mu\nu}F^{\mu\nu} \pm \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu} + F_{\mu\nu}F^{\mu\nu}\right) \geq 0 \end{aligned}. \tag{2.29}$$

From ((2.28) and (2.29) we see that

$$S[A] \geq \frac{1}{8}\left|\int d^4x\ \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right| = \frac{|I(U)|}{3N}. \tag{2.30}$$

From this inequality, it is clear that the lower bound of the action is reached only if $F_{\mu\nu}$ is self-dual or anti-self-dual

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{2.31}$$

Thus, any gauge field solution to the integral (2.28) is a minimum of the action. Belavin, Polyakov, Schwarz, and Tyupkin (BPST) found an instanton gauge configuration that solves (2.28) which takes the form in Euclidean space [5]

$$A_\mu(x) = \left(\frac{|\mathbf{x}|^2}{|\mathbf{x}|^2+\rho^2}\right)U^{-1}(\hat{x})\partial_\mu U(\hat{x}) \tag{2.32}$$

where ρ is the size radius of instanton, and $U(\hat{x})$ is an element of $SU(2) \subset G$ with

$$U(\hat{x}) = \left(\frac{x_4 + 2i\mathbf{x}\cdot\tau}{|\mathbf{x}|^2}\right). \tag{2.33}$$

and τ being the Pauli matrices. This solution satisfies the asymptotic behaviour of gauge field $A_\mu(x)$, and $U(\hat{x})$ is the same mapping as $g(x)$ in (2.15). This field configuration is localised at $x = 0$ with size ρ. A solution for a general winding number can be found by superimposing BPST instantons located far away from each other.

Therefore, using equations (2.18)-(2.20) and (2.30), we see that for a general winding number, the action and the Euclidean integral expressions become

$$\begin{aligned} S[A] &= 8\pi^2 n \\ \int d^4x \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} &= 64\pi^2 n \end{aligned} . \tag{2.34}$$

These results are valid for a normalization where we have absorbed a factor of $1/g$ into $F_{\mu\nu}$, so with a coupling constant g, the finite action becomes $S[A] = \frac{8\pi^2 n}{g^2}$. Consequently, this action contributes to the Euclidean path integral suppression by a factor $\exp\left(-\frac{8\pi^2 n}{g^2}\right)$.

The Standard Model is invariant under the gauge group $G = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. It is also invariant under accidental global baryon and lepton number conservation symmetries. These accidental symmetries are not violated at tree-level diagrams and higher orders in perturbation theory. Nevertheless, it is possible to see how the baryon and lepton numbers are not conserved within the Standard Model if one takes into account non-perturbative processes. In 1976 't Hooft realized how non-perturbative instanton field configurations with non-zero winding numbers give rise to processes that violate baryon and lepton number conservation. [22] However, the probability of such processes is exponentially suppressed by $\exp\left(-\frac{8\pi^2 n}{g^2}\right)$, which for electroweak processes is of order $\sim 10^{-170}$.

Lastly, one can now view the theorem (2.9) from another perspective. Consider our gauge field theory in the space of topology $\mathbf{S}^3 \times R$. Here, $\mathbf{S}^3$ is a closed 3-dimensional manifold, and in Euclidean coordinates, R is parametrised by the Euclidean time t_E. We can always associate a charge with the axial current: $Q_A = \int_{\mathbf{S}^3} {J_A}^0$. One may now integrate the anomaly equation (1.36) to find

$$\Delta Q_A = Q_A|_{t_E=+\infty} - Q_A|_{t_E=-\infty} = \frac{e^2}{16\pi^2} \int d^4x_E \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}. \tag{2.35}$$

The left-hand side of (3.35) clearly is integer-valued, and thus the right-hand side is also integer-valued because of the topology of the gauge field. This connection to topology of the ABJ anomaly explains why the anomaly equation (1.37) is 1-loop exact and does not change under higher-order loop corrections.

Chapter 3

Wess-Zumino-Witten term

We argued that at low energies, the dynamics of Goldstone modes can be described by the chiral lagrangian (1.54). Following the same symmetry transformation arguments as in Section 1.6, we could add higher-order derivative terms to our theory that would capture the corrections for interactions as we go to higher energies [14]. It turns out that we could include one more term that would capture the anomalies in the current algebra context.

3.1 Extra symmetry

We built the chiral lagrangian (1.54) by incorporating all the relevant symmetries of QCD. The problem, however, is that (1.54) possesses an extra discrete symmetry that is not a symmetry of QCD.

The chiral lagrangian is invariant under Poincaré transformations and the charge conjugation: $U \mapsto U^T$. It is also invariant under the time reversal $t \mapsto -t$ and the change of sign of bosons: $U \mapsto U^{-1} \Rightarrow \pi^a \mapsto -\pi^a$. We see that it counts the bosons mod 2 in (1.51) and we will denote it as $(-1)^{N_B}$ where N_B is the number of bosons. Lastly, (1.54) is invariant under the naive parity transformation, P_0: $\mathbf{x} \mapsto -\mathbf{x}$ with $t \mapsto t$ and $U \mapsto U$.

It turns out that $(-1)^{N_B}$ is not a symmetry of QCD. In QCD, the parity transformation acts as $\mathbf{x} \mapsto -\mathbf{x}$ with $t \mapsto t$ and $U \mapsto U^{-1}$. In terms of the chiral lagrangian symmetries, this acts as $P = P_0(-1)^{N_B}$. The QCD theory is invariant under the total P symmetry, but not under separate P_0 and $(-1)^{N_B}$ symmetries. For example, the process $K^+ + K^- \mapsto \pi^+ + \pi^0 + \pi^-$ violates P_0 and $(-1)^{N_B}$ individually, but not the total symmetry P. It is therefore natural to ask how to modify the chiral lagrangian in order to remove the redundant P symmetry.

Applying Euler - Lagrange equations to the chiral lagrangian, one finds the equation of motion

$$\frac{1}{2}{f_\pi}^2\partial_\mu\left(U^\dagger\partial^\mu U\right)=0. \tag{3.1}$$

We could add to this equation a unique extra term that would preserve P and violate P_0. Moreover, requiring Lorentz invariance and violation of P_0, this extra term should contain antisymmetric Levi - Civita symbol $\epsilon^{\mu\nu\alpha\beta}$. To correctly describe low energy physics, we require the smallest possible number of derivatives of U. Considering all of the above requirements, one could add to (3.1) the following extra term

$$\lambda\epsilon^{\mu\nu\alpha\beta}U^\dagger\left(\partial_\mu U\right)U^\dagger\left(\partial_\nu U\right)U^\dagger\left(\partial_\alpha U\right)U^\dagger\left(\partial_\beta U\right)=0 \tag{3.2}$$

with λ being a dimensionless constant. However, the problem arises when one starts to look for a Lagrangian from which (3.2) can be derived. A naive guess would be to add the 4 - dimensional pseudoscalar of the form $\lambda\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left(U^\dagger\left(\partial_\mu U\right)U^\dagger\left(\partial_\nu U\right)U^\dagger\left(\partial_\alpha U\right)U^\dagger\left(\partial_\beta U\right)\right)$, but this vanishes because $\epsilon^{\mu\nu\alpha\beta}$ is antisymmetric and the trace is cyclic. Nevertheless, it turns out that there is a way to construct a Lagrangian for this extra term.

3.2 Magnetic Monopole Analogy

Let us consider a particle of mass m moving on a two-dimensional sphere of unit radius. The equation of motion for this system is $m\ddot{x}_i+mx_i\left(\sum_k \dot{x_k}^2\right)=0$ with a constraint $\sum_i {x_i}^2=1$. The Lagrangian of this system is $\mathcal{L}=\int dt\frac{m}{2}\dot{x}_i^2$ and together with the equation of motion is invariant under separate time reversal $t\mapsto -t$ and parity $x_i\mapsto -x_i$ symmetries. If we want to reduce this symmetry and make the equation of motion invariant under the combined parity and time-reversal symmetry, the simplest choice would be

$$m\ddot{x}_i+mx_i\left(\sum_k \dot{x_k}^2\right)=\alpha\epsilon_{ijk}x_j\dot{x}_k \tag{3.3}$$

where α is a dimensionless coupling constant. If we try to construct the Lagrangian for (3.3) we run into a problem again since there is no obvious term that would reproduce the right-hand side of (3.3). A naive guess would be to add $\epsilon_{ijk}x_ix_j\dot{x}_k$ to the Lagrangian, but this is zero since the Levi - Civita symbol is antisymmetric and permutations of $x_ix_j\dot{x}_k$ are symmetric. Luckily, this problem has a well-known solution. The right-hand side of (2.3) can be understood as the Lorentz force acting on an electric charge and interacting with a magnetic monopole charge located at the centre of a 3 - dimensional sphere. Introducing a gauge potential $A_i(x)$, such that $\nabla\times\mathbf{A}=\frac{\mathbf{x}}{|x|^3}$, one can write the action for a particle as

$$S = \int_\gamma \left(\frac{1}{2} m {x_i}^2 + \alpha A_i(x) \dot{x}_i \right) dt \tag{3.4}$$

where γ is the worldline of a particle. This approach is problematic because the gauge potential in (3.4) suffers from a Dirac string singularity and breaks the $SO(3)$ rotational symmetry, which is preserved in the equation of motion (3.3). Let us explore this in quantum mechanics and write down the path integral for the gauge potential part of (3.4)

$$\exp\left(iS_{A_i}\right) = \exp\left(i\alpha \int_\gamma A_i dx_i\right) \tag{3.5}$$

where we consider integration over γ to be a closed particle orbit path. Using the Gauss's law, we can write (3.5) in terms of the magnetic flux through D

$$\exp\left(i\alpha \int_\gamma A_i dx_i\right) = \exp\left(i\alpha \int_D F_{ij} d\Sigma^{ij}\right) \tag{3.6}$$

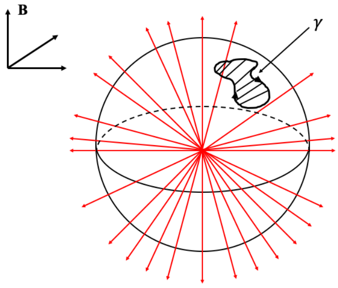
where D is a two-dimensional disc and in Figure 3.1a γ is the boundary of disc D: $\partial D = \gamma$. We could now try to use the path integral in terms of the field strength F_{ij}, but we run into a problem. The curve $\gamma' = -\gamma$ in Figure 3.1b bounds another surface disc D' and there is an ambiguity in the choice between D and D'. For the path integral to be well-defined, we require these both choices to give the same result

$$\exp\left(i\alpha \int_D F_{ij} d\Sigma^{ij}\right) = \exp\left(-i\alpha \int_{D'} F_{ij} d\Sigma^{ij}\right) \tag{3.7}$$

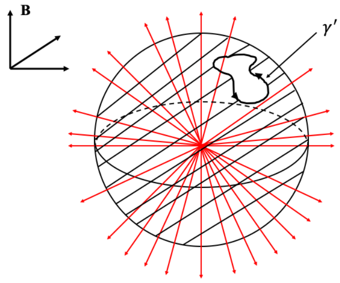
where we get a minus sign on the right-hand side because we choose γ bounding D to have the opposite direction to γ' which bounds D'. Stitching both D and D' discs into a whole 2-sphere $\mathbf{S}^2$, the condition (3.7) in the Feynman path integral can be written as

$$\exp\left(i\alpha \int_{D+D'} F_{ij} d\Sigma^{ij}\right) = \exp\left(i\alpha \int_{\mathbf{S}^2} F_{ij} d\Sigma^{ij}\right) = 1. \tag{3.8}$$

However, the flux through a sphere is equal to the total charge inside and we have another condition $\int_{\mathbf{S}^2} F_{ij} d\Sigma^{ij} = 4\pi$. Therefore, to satisfy (3.8), α has to be an integer or half-integer. This is the Dirac quantization law for the product of electric and magnetic charges.



(a) Integrating over the path γ. D is the dashed surface area.



(b) Integrating over the path γ'. D' is the dashed surface area.

Fig. 3.1 A particle orbit path over 2 - sphere.

3.3 Quantization of Wess - Zumino Action

Having discussed the magnetic monopole model, let us now return to our primary problem. Initially, we had an action for a point particle that has a one-dimensional worldline in spacetime, and thus the action for this particle is the integration along the worldline. We extended our model to a two - dimensional string (3.4), which had a two-dimensional worldsurface in spacetime, and the action for this string was the integration over its worldsurface. Now we imagine our four-dimensional spacetime $\mathcal{M}$ for our field $U(x)$ defined in (1.51). By the analogy to the one - dimensional orbit γ in Figure 3.1 which is the boundary to the two - dimensional disc D, we introduce a five-dimensional ball Q, such that $\partial Q = \mathcal{M}$.

We extend Goldstone fields $U(x)$ over $\mathcal{M}$ to a unitary matrix $U(y)$, defined in 5-dimensional ball Q with coordinates $\{x^\mu, s\}$. Now, we consider the following Wess-Zumino term for $U(y)$

$$\omega(y) = -\frac{i}{240\pi^2}\epsilon^{ijklm}\mathrm{Tr}\left(U^\dagger\frac{\partial U}{\partial y_i}U^\dagger\frac{\partial U}{\partial y_j}U^\dagger\frac{\partial U}{\partial y_k}U^\dagger\frac{\partial U}{\partial y_l}U^\dagger\frac{\partial U}{\partial y_m}\right) \tag{3.9}$$

with i, j, l, m running over the coordinates $\{x^\mu, s\}$. Constructed this way, $\omega(y)$ is invariant under the $SU(N_f)_L \otimes SU(N_f)_R$ transformations. Moreover, we can write the integral for $\omega(y)$

$$\Gamma(U) = \int_Q d^5y\omega(y) \tag{3.10}$$

which depends only on the values taken by $U(x)$ fields in spacetime. To check this, we apply the same *Cartan-Maurer invariant* variation arguments to (3.9). In the context of the effective Lagrangian, our matrix manifold is identified with $SU(3)$ due

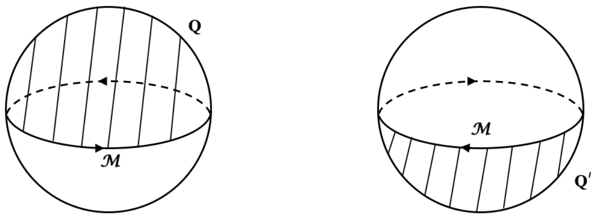


Fig. 3.2 Integrating over 5-dimensional balls Q and Q' with spacetime $\mathcal{M}$ at the boundary.

to Goldstone's theorem. Analogous to (2.11), we get that $\omega(y)$ changes by a derivative

$$\delta\omega(y) = -\frac{i}{48\pi^2}\frac{\partial}{\partial_m}\mathrm{Tr}\left[U^\dagger\frac{\partial U}{\partial y_i}U^\dagger\frac{\partial U}{\partial y_j}U^\dagger\frac{\partial U}{\partial y_k}U^\dagger\frac{\partial U}{\partial y_l}U^\dagger\delta U\right] \tag{3.11}$$

and so a change $\delta U(y)$ that does not affect $U(y)$ in spacetime also does not affect the integral $\int_Q d^5y\ \omega(y)$. Therefore, we may include the Wess-Zumino term in the effective Lagrangian as

$$\mathcal{L}_{eff} = \frac{f_\pi{}^2}{4}\mathrm{tr}\left(\partial^\mu U^\dagger\partial_\mu u\right) + n\Gamma(U) \tag{3.12}$$

where n is some arbitrary coefficient.

Just as in the magnetic monopole model, there is an ambiguity in the way we include the Wess-Zumino term in the action. We may think of our 5-dimensional ball Q as a half of the 5-sphere $\mathbf{S}^5$ and the spacetime $\mathcal{M}$ as the boundary between Q and the other half Q' in Figure 3.2. Since $\mathcal{M}$ is also a boundary of Q', we could have also included the Wess-Zumino term into the action as

$$I'(U) = -n\int_{Q'} d^5y\omega(y) \tag{3.13}$$

with a negative sign appearing since the boundary $\mathcal{M}$ at Q' is of the opposite orientation. It is not possible to require that $I(U) = I'(U)$ for Goldstone boson fields unless setting $n = 0$. But since the Wess-Zumino action enters the path integral as $e^{iI(U)}$, and if we want the path integral to be unaffected by the difference between $I(U)$ and $I'(U)$, it is therefore necessary to require that

$$I(U) - I'(U) = n\int_{\mathbf{S}^5} d^5y\omega(y) = 2\pi k, \quad k \in Z \tag{3.14}$$

where the 5-sphere is $\mathbf{S}^5 = Q + Q'$.

We now need a classification of mappings $\mathbf{S}^5 \mapsto SU(3)$. Such mappings are characterized by the fifth homotopy group $\pi_5(SU(3)) = Z$. The normalised integral for $\omega(y)$ over a basic 5-sphere $\mathbf{S}_0$ for such mappings is $\int_{\mathbf{S}_0} d^5y\omega(y) = 2\pi$. Moreover, we may use Bott's [8] periodicity theorem that allows us to topologically treat any 5-sphere in $SU(3)$ as a multiple of the basic 5-sphere $\mathbf{S}_0$. So, it follows that after normalisation, the coefficient n in front of the Wess-Zumino term entering the Lagrangian (3.12) must be an integer. Because of this n coefficient condition found by Witten, the $n\Gamma$ term is also known as the Wess-Zumino-Witten (WZW) term [24].

This WZW term is invariant under the $SU(N_f)_L \otimes SU(N_f)_R$ transformations, and one can see the physical consequences of (3.12) in the following steps. First, infinitesimally expanding the U field in terms of pions π like in (1.66), one gets a relation to the first order

$$U^\dagger \partial_i U \approx \frac{2i}{f_\pi} \partial_i \pi + \dots \tag{3.15}$$

and plugging it into (3.9) gives

$$\begin{aligned} \omega_{ijklm} d\Sigma^{ijklm} &= d\Sigma^{ijklm} \frac{2}{15\pi^2 {f_\pi}^5} \text{tr}\left(\partial_i\pi\partial_j\pi\partial_k\pi\partial_l\pi\partial_m\pi\right) = \\ & d\Sigma^{ijklm} \frac{2}{15\pi^2 {f_\pi}^5} \partial_i \left(\text{tr}\left(\pi\partial_j\pi\partial_k\pi\partial_l\pi\partial_m\pi\right)\right) \end{aligned} \tag{3.16}$$

where in the last equality the antisymmetry property of $d\Sigma^{ijklm}$ was used. Written in this way, the action $n\Gamma$ is the integral of a total divergence over a five-sphere Q, and using Stokes' theorem, we can rewrite this integral over the boundary of Q which is the spacetime $\mathcal{M} = \partial Q$. We have then

$$n\Gamma = \frac{2n}{15\pi^2 {f_\pi}^5} \int_{\mathcal{M}} \epsilon^{\mu\nu\alpha\beta} \text{tr}\left(\pi\partial_\mu\pi\partial_\nu\pi\partial_\alpha\pi\partial_\beta\pi\right) d^4x. \tag{3.17}$$

This 4-dimensional integral entering the Lagrangian in (3.12) is no longer $SU(3)_L \otimes SU(3)_R$ invariant. Nonetheless, since this expression came from the 5-dimensional WZW term, the $SU(3)_L \otimes SU(3)_R$ symmetry must be hidden. In the context of QCD, the effective Lagrangian with (3.17) provides a description for the low energy limit of the decay $K^+ + K^- \mapsto \pi^+ + \pi^0 + \pi^-$. Therefore, the inclusion of the WZW term in the effective Lagrangian removes the redundant $(-1)^{N_B}$ symmetry.

3.4 Relation to the *ABJ* Anomaly

It is natural for one to ask what is the value of integer n related to in the QCD context. The simplest way to see this is to gauge a $U(1)$ subgroup of $SU(N_f)_{diag} \subset SU(N_f)_L \otimes SU(N_f)_R$, i.e. coupling U to electromagnetism. For $N_f = 3$, we introduce a charge matrix $Q = \text{diag}[\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}]$ and promote derivatives in the total chiral action (3.12) to covariant derivatives: $D_\mu U = \partial_\mu U - ieA_\mu[Q, U]$. In the case of the Γ term, it is tempting to do straightforward covariant derivative promotion $\partial_\mu U \to D_\mu U$, but it depends on 5 dimensions.

We first consider infinitesimal local transformations $\delta U = i\alpha(x)[Q, U]$, and applying to the term $U^\dagger \partial U$, we get $\delta \left(U^\dagger \partial U\right) = i\alpha(x) + i\partial\alpha U^\dagger [Q, U]$. Using these transformations in (3.9), one finds that the variation of $\omega_{\mu\nu\alpha\beta\tau}$ has terms of order $\partial^i \alpha$, with $i = 0, \ldots, 5$. Now, we may use the results of the discussion of the *Cartan-Maurer invariant*: only $i = 1$ survives, because for $i = 0$ $\partial^i \alpha$ vanish by the cyclicity of the trace, and for $i = 2, \ldots, 5$ $\partial^i \alpha$ vanish by the antisymmetry of the anti-symmetric symbol. Using the identity $U^\dagger \partial_\mu U = -(\partial_\mu U^\dagger) U$ in the 5 - form $\omega_{\mu\nu\alpha\beta\tau}$ variation, one finds $\delta\omega_{\mu\nu\alpha\beta\tau} = \partial_\mu(\alpha) J^\mu$ and (3.10) transforms as

$$\Gamma \mapsto \Gamma' = \Gamma - \int \partial_\mu(\alpha) J^\mu d^4x \tag{3.18}$$

where the current resulting from Noether's theorem is

$$J^\mu = \frac{1}{48\pi^2} \epsilon^{\mu\nu\alpha\beta} \text{tr} \left(Q\Big[\partial_\nu U U^\dagger + \partial_\alpha U U^\dagger + \partial_\beta U U^\dagger + U^\dagger \partial_\nu U + U^\dagger \partial_\alpha U + U^\dagger \partial_\beta U\Big]\right). \tag{3.19}$$

To make the Lagrangian for Γ gauge - invariant, we add the coupling to the photon field A_μ, since under infinitesimal gauge transformations, $\delta A_\mu = \frac{1}{e}\partial_\mu \alpha$, and so $\Gamma \mapsto \Gamma' = \Gamma - e \int A_\mu J^\mu d^4x$.

However, this expression is still not gauge-invariant, because J^μ is not. Witten used a trial and error Noether method to find the remaining term for the gauged WZW action [24]. However, in this method, the parity conservation was not imposed properly, as mentioned in [10]. This led to efforts of several groups [18], [19], [19] to find a general method for gauging the topological WZW term for arbitrary gauge subgroups of $SU(N)_L \otimes SU(N)_R$. This requires differential geometry techniques involving the construction of gauge-invariant (2n - 1) - forms which satisfy the Abelian anomalous Ward identities in (2n + 2) - dimensional space. Consequently, it turns out the gauge-invariant WZW action coupled to electromagnetism may be written as

$$\begin{aligned}\tilde{\Gamma}(U, A_\mu) = \Gamma(U) - e \int A_\mu J^\mu d^4x \\ + \frac{ie^2}{24\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \left(\partial_\mu A_\nu\right) A_\alpha \\ \times \mathrm{tr}\left(Q^2 \partial_\beta U U^\dagger + Q^2 U^\dagger \partial_\beta U + \frac{1}{2} Q \partial_\beta U Q U^\dagger - \frac{1}{2} Q U Q \partial_\beta U^\dagger\right).\end{aligned} \tag{3.20}$$

Now the gauge-invariant version of the low-energy action is

$$\mathcal{L} = \frac{{f_\pi}^2}{4} \int \mathrm{tr}\left(D_\mu U^\dagger D^\mu U\right) d^4x + n\tilde{\Gamma}. \tag{3.21}$$

To see how this helps us determine n, let us expand the action (3.21) in terms of the pion fields π. Remember that we are working with $N_f = 3$ quarks, and using the charge matrix Q values, one finds the order e^2 piece

$$\mathcal{L} = \frac{ne^2}{96\pi^2 f_\pi} \pi^0 \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \tag{3.22}$$

This agrees with the axial anomaly, which we have already seen in this form in Section 1.9: when $n = N_c$, the number of colours [24]. This is a surprising result since until now the chiral Lagrangian appeared to be independent of the gauge group $SU(N_c)$. We now see that there is some memory of the underlying $SU(N_c)$ gauge group that survives as the integer coefficient in (3.21).

Conclusions and Outlook

We have investigated the phenomenon of chiral symmetry breaking, which is caused by the order parameter of the quark condensate. While we have presented an indirect argument on how this is related to confinement, the analytic calculation of the quark condensate remains an open problem.

We have also witnessed how the ABJ anomaly enters the path integral in the measure for fermionic fields. We investigated the derivation of the anomaly and its link to topological instanton field configurations in Chapter 2. This helped us understand why the anomaly equation is 1-loop exact without the need to calculate higher-order loop diagrams.

We have also discussed the construction of an effective low-energy theory for pions by considering relevant symmetry transformations. The utility of current algebra in the chiral Lagrangian turned out to be successful in predicting certain pion interaction processes to the lowest order (i.e. at two-derivative order we denote it as $\mathcal{L}_2$). Higher-order derivative terms could be included in the chiral Lagrangian by considering the symmetry requirements discussed in Section 1.6, and thus the chiral Lagrangian could be generalized to a series of increasing order derivatives as $\mathcal{L}_{chiral} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 \dots$

Lastly, in Chapter 3, we discussed the extension of the chiral Lagrangian with the WZW term, which turned out to be successful in describing the $K^+ + K^- \mapsto \pi^+ + \pi^0 + \pi^-$ decay and capturing the ABJ anomaly. While we have briefly described the framework of quantisation and the method of gauging the WZW term in the chiral Lagrangian, we have not mentioned possible static Soliton field configurations for this action. These field configurations are known as Skyrmions [20], and the idea that these Solitonic solutions to the effective chiral Lagrangian can be understood as baryons has a long history. An overview of this framework can be found in [23] and [10].

References

[1] Adler, S. L. (1969). Axial-Vector Vertex in Spinor Electrodynamics. *Phys. Rev.*, 177:2426–2438.

[2] Adler, S. L. and Bardeen, W. A. (1969). Absence of higher-order corrections in the anomalous axial-vector divergence equation. *Phys. Rev.*, 182:1517–1536.

[3] Atiyah, M. F. and Singer, I. M. (1984). Dirac operators coupled to vector potentials. *Proceedings of the National Academy of Sciences of the United States of America*, 81:2597–2600.

[4] Bardeen, W. A. (1969). Anomalous Ward Identities in Spinor Field Theories. *Phys. Rev.*, 184:1848–1859.

[5] Belavin, A., Polyakov, A., Schwartz, A., and Tyupkin, Y. (1975). Pseudoparticle solutions of the Yang-Mills equations. *Physics Letters B*, 59:85–87.

[6] Bell, J. S. and Jackiw, R. (1969). A PCAC Puzzle: $\pi \mapsto 2\gamma$ in the σ-model. *Nuovo Cim.*, 47.

[7] Bjorken, J. D. and Drell, S. D. (1964). *Relativistic Quantum Mechanics*. McGraw-Hill, New York.

[8] Bott, R. and Seeley, R. (1978). Some remarks on the paper of callias. *Commun.Math. Phys.*, 62:235–245.

[9] Coleman, S., Wess, J., and Zumino, B. (1969). Structure of Phenomenological Lagrangians. I. *Phys. Rev.*, 177:2239–2247.

[10] Curtis G. Callan, E. W. (1984). Monopole catalysis of Skyrmion decay. *Nuclear Physics B*, 239:161–176.

[11] Fujikawa, K. (1979). Path-Integral Measure for Gauge-Invariant Fermion Theories. *Phys. Rev. Lett.*, 42:1195–1198.

[12] G. Hooft, e. a. (1980). Naturalness, Chiral Symmetry, and Spontaneous Chiral Symmetry Breaking. *Recent Developments in Gauge Theories*, 59(4):385–403.

[13] Gasser, J. and Leutwyler, H. (1984). Chiral Perturbation Theory to One Loop. *Annals of Physics*, 158:142–210.

[14] Gasser, J. and Leutwyler, H. (1985). Chiral perturbation theory: Expansions in the mass of the strange quark. *Annals of Physics*, 250:465–516.

[15] Gell-Mann, M., Oakes, R. J., and Renner, B. (1968). Behavior of Current Divergences under $SU(3) \times SU(3)$. *Phys. Rev.*, 175:2195–2199.

[16] Georgi, H. (1984). *Weak Interactions and Modern Particle Theory.* Benjamin/Cummings, Menlo Park.

[17] J. Goldstone, A. Salam, S. W. (1962). Broken Symmetries. *Phys. Rev.*, 127:965–3970.

[18] Kuang-chao, C., Han-ying, G., Ke, W., and Xing-chang, S. (1988). On the gauge invariance and anomaly-free condition of the Wess-Zumino-Witten effective action. *Physics Letters B*, 134:67–69.

[19] Mañes, J. (1985). Differential geometric construction of the gauged Wess-Zumino action. *Nuclear Physics B*, 250:369–384.

[20] Skyrme, T. H. R. (1961). A non-linear field theory. *Proc. Roy. Soc.*, 127:127–138.

[21] Steinberger, J. (1949). On the Use of Subtraction Fields and the Lifetimes of Some Types of Meson Decay. *Phys. Rev.*, 76:1180–1186.

[22] 't Hooft, G. (1976). Computation of the quantum effects due to a four-dimensional pseudoparticle. *Phys. Rev. D*, 14:3432–3450.

[23] Witten, E. (1983a). Current algebra, baryons, and quark confinement. *Nuclear Physics B*, 223:433–444.

[24] Witten, E. (1983b). Global aspects of current algebra. *Nuclear Physics B*, 223:422–432.