Quantum Entanglement in Gauge Theories

Contents

1 Introduction

1.1 What is entanglement entropy?

Entropy measures the lack of knowledge about the state of a physical system. In quantum mechanics and quantum field theory, it is possible to have complete knowledge of the quantum state of the whole universe, or any closed system, and yet not know the exact state of its smaller subsystem described by a local density matrix ρ_A. This is because quantum entanglement between local degrees of freedom in different regions introduces non-local correlations, which cannot be fully determined using only measurements on ρ_A. From this, a particular kind of entropy emerges, called *entanglement entropy*. It measures the strength of these non-local correlations.

1.2 Why is entanglement entropy important?

Since first definitions of entanglement entropy, entanglement entropy has gained attention for numerous reasons. Here, the emphasis will be on its applications to quantum many-body physics, and quantum computation.

1. It has ties to **quantum many-body systems**, especially to critical phenomena and phase transitions [33, 18]. For instance, it has been shown that a gapped quantum many-body spin system on a lattice satisfies an area law for entanglement entropy if and only if any other state in the same phase also satisfies an area law [31]. Another relation is that quantities dependent on the entropy were shown to coincide with critical points of lattice systems [9]. Additionally, entanglement entropy can be used to detect and describe topological phases of matter [16].

2. It is widely used in **quantum computation and information** areas. For example, it coincides with the number of Bell pairs needed to distill a state [21], and measures the information capacity of quantum channels [3]. Note that those systems are usually not based on gauge theories.

3. Combining topological matter with quantum computing leads to **topological quantum computation**. There, the information is stored in quasiparticle excitations called *anyons*, which emerge in topologically ordered states. The information stored in anyons is stored in their global properties, and is thus robust to small perturbations [16, 15]. The simplest example is the *toric code* [15] - a quantum error correction code defined on a $\mathbb{Z}_2$-gauged lattice. Entanglement entropy may be a good way to quantify the amount of information stored and processed in topological algorithms, thus quantifying their complexity and security.

Studying entanglement entropy in gauge theories is thus justified by its connection to topological matter and topological quantum information.

However, in gauge theories additional difficulties arise due to the fact that the underlying Hilbert space of available states is **not a tensor product** of local Hilbert spaces. As the original definition of entanglement entropy assumes a tensor product Hilbert space, it is necessary to redefine entanglement entropy for gauge theories. Comparing and justifying these new definitions is the main motivation for this work.

1.3 Formal definition of entanglement entropy

Let $|\psi\rangle$ be a vector in the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of all quantum states of a theory. Every pure quantum state $|\psi\rangle$ can be described by its density matrix ρ.

$$\rho = |\psi\rangle \langle\psi| . \tag{1}$$

For mixed states, this generalises to

$$\rho = \sum_{i=1}^{\dim \mathcal{H}_{AB}} p_i |\psi_i\rangle \langle\psi_i| , \quad p_i \in [0,1], \quad \sum_i p_i = 1 \tag{2}$$

where p_i denotes the probability of the system being in state $|\phi_i\rangle$, and $|\psi_i\rangle$ form an orthonormal basis of $\mathcal{H}_{AB}$. This diagonal form always exists as ρ is Hermitian and positive semi-definite. The reduced density matrix on a subsystem $\mathcal{H}_A$ is given by:

$$\rho_A = \mathrm{Tr}_B(\rho), \tag{3}$$

where $\mathrm{Tr}_B(\rho)$ is the partial trace of ρ over $\mathcal{H}_B$. ρ_A fully describes the degrees of freedom accessible locally in $\mathcal{H}_A$. Finally, the entanglement entropy S_A between $\mathcal{H}_A$ and $\mathcal{H}_B$ is defined using the reduced density matrix,

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A). \tag{4}$$

As the reduced density matrix ρ_A is also Hermitian and positive semi-definite, it can be diagonalised. From the invariance of trace under a change of basis, we can express the entanglement entropy in terms of the eigenvalues λ_i of ρ_A:

$$S_A = -\sum_{\imath=1}^{\dim \rho_A} \lambda_i \log \lambda_i. \tag{5}$$

Entanglement entropy is a measure the strength of quantum correlations between $\mathcal{H}_A$ and $\mathcal{H}_B$. It is worth noting that it is not the only possible measure of entanglement strength - other measures exist and are used in various context. More information about this can be find in work about *entanglement monotones*, e.g. [34].

Entanglement entropy has a few key properties frequently used in proofs and calculations, which would ideally be preserved in any new definition. Firstly, it is symmetric - for any region divided into 2 subregions A and B, we have $S_A = S_B$. It is also subadditive:

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B), \tag{6}$$

and in a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, strongly subadditive:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}). \tag{7}$$

1.4 Entanglement entropy in field theories: a general recipe

A general recipe to study entanglement entropy in particular theories and systems is as follows.

1. Define the physical space on which the system is described, e.g. $\mathbb{R}^{1,3}$, or a 2D discrete lattice.

2. Define the global Hilbert space, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

3. Find the particular quantum state of the system, $\rho_{AB} \in \mathcal{H}_{AB}$, for which you want to calculate the entropy. This can be, and often is, a ground state of a particular Hamiltonian which then has to be given.

4. Calculate the entanglement entropy using Eq. (4).

Despite their apparent simplicity, each step may introduce technical difficulties into the calculation, for example:

1. If the studied physical space is continuous, decomposition into tensor product is often not straightforward or not possible due to the infinite degrees of freedom of the theory. The difficulties in calculating entanglement entropy the continuum limit are an ongoing area of research, which will not be presented in this essay.

2. There must be no global constraints present in order to partition the Hilbert space into a direct product of local subspaces - hence the difficulties in gauge theories, but also in other cases, e.g. in effective descriptions of quantum mechanical scenarios where global number of particles is conserved [26].

3. The Fock space of QFTs is complicated, and often it is not realistic to know the precise state of the system. To circumvent this, particular examples of states or statistical ensembles are studied.

4. Eq. (4) includes a logarithm of a density matrix, which can be difficult to calculate, e.g. if ρ_{AB} is expressed as a path integral. This can in many cases be solved by the replica trick, explained below in Section 1.5.

1.5 The replica trick

The replica trick is a technique which allows to compute entanglement entropy without computing density matrix logarithms. Starting from Eq. (4) on a general ρ, we get:

$$S = -\mathrm{Tr}(\rho \log \rho) \tag{8}$$

$$= -\mathrm{Tr}\left(\sum_i^N p_i \left|\psi_i\right\rangle \left\langle\psi_j\right| \log\left(\sum_j^N p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|\right)\right) \tag{9}$$

$$= -\mathrm{Tr}\left(\sum_i^N p_i \left|\psi_i\right\rangle \left\langle\psi_j\right| \left(\sum_j^N \log p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|\right)\right) \tag{10}$$

$$= -\mathrm{Tr}\left(\sum_{i,j}^N p_i \left|\psi_i\right\rangle \delta_{i,j} \log p_j \left\langle\psi_j\right|\right) \tag{11}$$

$$= -\sum_i^N p_i \log p_i \tag{12}$$

$$= -\lim_{n\to 1} \sum_i^N \frac{d}{dn} p_i^n \tag{13}$$

$$= -\lim_{n\to 1} \frac{\partial}{\partial n} \rho^n \tag{14}$$

This reduces the problem of calculating the entanglement entropy to calculating ρ^n, with $n \in \mathbb{R}$. This method is often used as a reference to check other methods.

2 Entanglement entropy in non-gauge lattice theories

2.1 Basic setup and scaling laws

Consider a d-dimensional square grid. On each lattice site, i.e. vertex, there is a quantum-mechanical system with local degrees of freedom, with states taking on values $\left|\psi\right\rangle_v \in \mathcal{H}_v$. For simplicity, take $\mathcal{H}_v = \mathcal{H}$ to be the same for all vertices v. Label the set of all vertices $\{v\} = V$. Given any subset of vertices $A \in V$, its Hilbert space is defined as:

$$\mathcal{H}_A = \bigotimes_{a\in A} \mathcal{H}_a = \mathcal{H}^{\otimes |A|}. \tag{15}$$

Similarly, the Hilbert space of the complement $\bar{A}$ is just $\mathcal{H}^{\otimes|\bar{A}|}$. The entanglement entropy of such a system can be calculated using the usual definition in Eq. (4).

An important research question, which enables e.g. characterising phases of matter [16, 31], is: in such systems, how does the entanglement entropy scale? There are 3 main possibilities:

- An area-law scaling, proportional to the size of the boundary between regions A and B. It arises in ground states of systems with only local interactions, where only the local degrees of freedom around the boundary contribute to entanglement between A and B. A review of non-gauge systems following area laws can be found in [12].

- A logarithmic scaling with volume, usually observed in critical systems which can be described by conformal field theories [6].

- A volume-law scaling, followed by most possible states in a Hilbert space [4, 35].

Similar scaling laws will appear in gauge theories, using new definitions of entanglement entropy.

2.2 A warm-up example: 2 entangled qubits

First, let us try to calculate entanglement entropy of a specific system. We will use the explicit method - finding the exact density matrix and all its eigenvalues. This is often not possible in more complicated theories.

The simplest system to consider is a *Bell pair*, i.e. a pair of maximally entangled qubits - qubit A and qubit B, treated as subsystems. As there are no global constraints present, the underlying Hilbert space can be decomposed into a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\mathcal{H}_A = \mathcal{H}_B$, as they are identical qubits. A possible particular Bell state is:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{16}$$

whose density matrix is:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

The notation $|00\rangle$ is a shorthand for $|0\rangle_A \otimes |0\rangle_B$.

The reduced density matrix ρ_A is

$$\sum_{i=1}^{d_B} \langle i_B| \rho |i_B\rangle , \tag{18}$$

where $\{\langle i_B|\}$ is the basis for $\mathcal{H}_B$, here - $\{|0\rangle_B, |1\rangle_B\}$. Hence

$$\rho_A = \langle 0|_B \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)|0\rangle_B \tag{19}$$

$$+ \langle 1|_B \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)|1\rangle_B \tag{20}$$

$$= \frac{1}{2}\left(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A\right) \tag{21}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

The eigenvalues of ρ_A are just 2 copies of $\frac{1}{2}$. Therefore the entanglement entropy is:

$$S_A = -\sum_{i=1}^{2} \lambda_i \log \lambda_i = -2 \times \frac{1}{2}\log\frac{1}{2} = \log 2 \tag{23}$$

It turns out that this is the maximal value of entanglement entropy which any state of 2 qubits can have [22]. This is why we call it the *maximally entangled* state.

Contrary to the premise of emerging area laws, the calculation suggests an extensive entropy. This is because the dimension of ρ_A was, and will always be, the dimension d_A of $\mathcal{H}_A$ - here, ρ was a 2×2 matrix. Assuming that in higher dimensions ρ_A is proportional to identity (this is called a *maximally mixed* state), the entanglement entropy will be

$$S_A = -d_A \times \frac{1}{d_A} \log \frac{1}{d_A} = \log d_A \tag{24}$$

In case of an N-qubit system, the dimension d_A scales exponentially with number of qubits, so S_A is expected to grow linearly with N. For N Bell pairs, this directly leads to $S_A = N \log 2$, so the entanglement entropy is indeed extensive. In more complicated N-qubit states, there can be an area law emerging.

3 Lattice gauge theories

3.1 Difference from lattice spin systems

In non-gauge lattice spin systems, the degrees of freedom live on the **vertices** in the lattice, as we saw in Section 2.2. In lattice gauge theories, the degrees of freedom live on the **links** between lattice sites, or equivalently the edges of the graph. This was introduced in Wilson's original paper [36]. The fundamental variables of the model are the values of gauge fields, taking values in the gauge group G. Matter degrees of freedom can be added on each vertex. Below, we will first consider a pure gauge theory, i.e. one consisting only of configurations of gauge fields, with no matter present. In this section, we will follow the definitions and symbols widely used in papers about entanglement entropy, e.g. by Casini et al. [8] and Donnelly [10, 11].

3.2 Defining the graph

Consider a lattice gauge theory with a gauge group G, either a discrete group or a compact Lie group. Then, on the graph:

- to each vertex a, a group element $g_a \in G$ is assigned
- to each directed edge l, further called a link, a group element $U_l \in G$ is assigned; links can also be denoted (ab), meaning the link from node a to node b,
- for each directed edge, there exists an edge with opposite orientation connecting the same vertices, i.e. $\bar{l}$, to which the inverse group element $U_{\bar{l}} = U_l^{-1} \in G$ is assigned,
- each edge has a source vertex and a target vertex, denoted $s(l)$ and $t(l)$ respectively.

The variables U_l represent local values of the gauge fields. The sets of field values across all the links, $\{U_l\}$, are called the *field configurations*. The variables g_a are the local gauge transformation which can be performed on the fields U_l. Gauge transformations on the field configurations are performed in the following way:

$$U_l \to g_{t(l)} \circ U_l \circ g_{s(l)}^{-1} \tag{25}$$

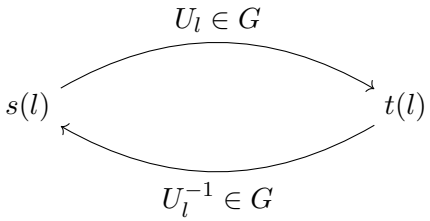

Figure 1: The simplest subgraph of any lattice gauge theory - a single link with a source vertex $s(l)$ and a target vertex $t(l)$.

3.2.1 What is the gauge-invariant Hilbert space?

Now, the aim is to find all possible gauge-invariant states of a given configuration $\{U_l\}$. The set of gauge-invariant states of the system must be a subset of $\mathcal{H}$, i.e. the space of all possible linear functionals on $[U]$:

$$|\Psi\rangle \equiv \Psi[U], \qquad \text{where } U = \{U_l\}. \tag{26}$$

However, not every state in $\mathcal{H}$ is gauge invariant. Because U and U' are physically equivalent if they are related by a gauge transformation, we have to construct a space of functionals Ψ on *equivalence classes* of field configurations under all possible gauge transformations, i.e. we require

$$\Psi[U] = \Psi[U']. \tag{27}$$

We want to impose gauge invariance *locally*. That means, if the gauge transformation at a single vertex a is performed according to Eq. (25) for all links connected to a, the state will be unchanged. Suppose there is a variable g_a assigned to the vertex a, and a is connected to exactly 4 links on a lattice: $U_{l_1}, U_{l_2}, U_{l_3}, U_{l_4}$, where the links are directed *from* a, i.e. $s(l_i) = a \ \forall_{i\in\{1,2,3,4\}}$. Then, the condition imposing gauge invariance on the vertex a is:

$$\Psi[\{U_1, ..., U_{l_1}, U_{l_2}, U_{l_3}, U_{l_4}, ..., U_n\}] = \Psi[\{U_1, ..., g_a U_{l_1}, g_a U_{l_2}, g_a U_{l_3}, g_a U_{l_4}, ..., U_n\}]. \tag{28}$$

Let us introduce a more concise notation for this criterion. Denote an operator $\hat{L}_g^l$ on $\mathcal{H}$ as:

$$(\hat{L}_g^l \Psi)[U_1, \ldots, U_N] = \Psi[U_1, \ldots, gU_l, \ldots, U_N]. \tag{29}$$

This just multiplies any link variable U_l by a chosen group element g. This transformation is the same as multiplying the opposite link variable by the inverse element, i.e.

$$\hat{L}_a^{(ab)} = \hat{L}_{g^{-1}}^{(ba)} \tag{30}$$

Using the $\hat{L_g^l}$ operators, Eq. (28) can be written as:

$$(\prod_i \hat{L}_{g_a}^{l_i} \Psi)[U] = \Psi[U]. \tag{31}$$

The complete local gauge transformation operator $\hat{T}_{g_a}$ is defined as:

$$\hat{T}_{g_a} = \prod_b \hat{L}_g^{(ab)}, \tag{32}$$

where the product is over all vertices b connected to a. Finally, the **gauge invariance condition** is:

$$\Psi[U] \in \mathcal{H}, \qquad (\hat{T}_{g_a}\Psi)[U] = \Psi[U] \quad \forall g_a \in G. \tag{33}$$

The corresponding Hilbert space of gauge-invariant states will be denoted $\mathcal{H}_{ginv}$. In general, it does not allow for a tensor product decomposition.

Equivalently, if we consider the restriction of the operator $\hat{T}_{g_a}$ to the algebra on the gauge-invariant Hilbert space, $\hat{T}_{g_a} \in \mathcal{B}(\mathcal{H}_{ginv})$, it must be equal to the identity:

$$\hat{T}_{g_a} = \mathbb{1} \tag{34}$$

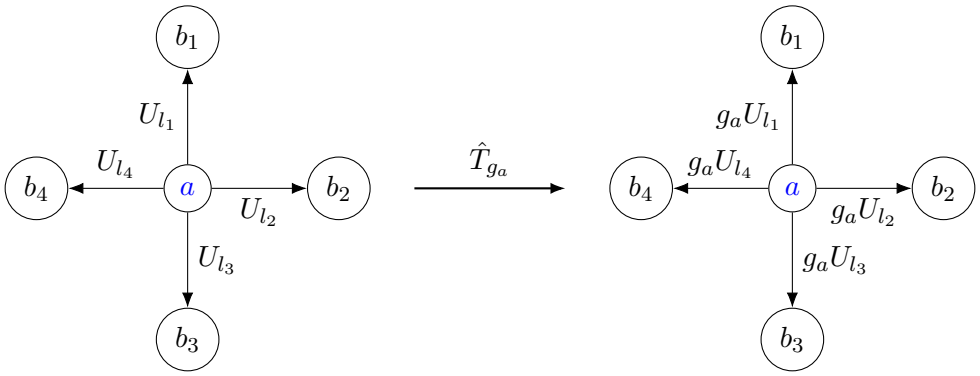

Figure 2: Gauge transformation by g_a. Note that $\Psi[U]$ must be taken of both configurations above for full mathematical correspondence with Eq. (25). Gauge-invariant states are unchanged by all such transformations.

3.3 What are the gauge-invariant observables?

For future usage in Section 4, we will need the set of all possible gauge-invariant observables. For simplicity, we first consider Abelian gauge groups.

The gauge-invariant algebra of observables is defined as the set of all operators $B \in \mathcal{B}(\mathcal{H})$ such that:

$$\Psi[U] \text{ is gauge invariant } \implies B(\Psi)[U] \text{ is gauge invariant.} \tag{35}$$

In fact, it is sufficient to find the **generating set** of the underlying operator algebra, i.e. set of operators $\mathcal{C}$ such that all others can be expressed as polynomials of $C \in \mathcal{C}$. This is because products of gauge-invariant operators are gauge-invariant from associativity, and linear combinations of gauge-invariant operators are gauge-invariant from linearity.

Firstly, notice that $\hat{L}_g^l$ **are gauge-invariant** for Abelian groups:

$$\begin{aligned}
\hat{T}_{g_a}(\hat{L}_h^{(ab)}\Psi)[U_{(ab)}] &= \hat{T}_{g_a}\Psi[hU_{(ab)}] \\
&= \Psi[ghU_{(ab)}] \\
&= \Psi[hgU_{(ab)}] \\
&= \hat{L}_h^{(ab)}(\hat{T}_{g_a}\Psi)[U_{(ab)}] \\
&= \hat{L}_h^{(ab)}(\Psi)[U_{(ab)}] \quad \text{from gauge-invariance of } \Psi.
\end{aligned} \tag{36}$$

However, $\hat{L}_g^l$ do not generate all possible gauge-invariant operators, so another set of operators is needed. Let $\hat{U}_l$ be the operator in $\mathcal{B}(\mathcal{H})$ such that:

$$(\hat{U}_l\Psi)[U] = U_l^r\Psi[U]. \tag{37}$$

As G is Abelian, there exists a non-trivial representation of G acting on a 1-dimensional vector space, which assigns a single number to each $g \in G$. This number is denoted U_l^r for $g = U_l$. $\hat{U}_l$ is the analogue of a position operator, as it does not change the underlying configuration $[U]$. For non-Abelian theories, an analogous operator independent of the 1-dimensional representation can be defined, see e.g. Eq. (139).

Importantly, $\hat{U}_l$ **are not gauge invariant**. Consider a graph with 2 vertices a, b:

$$\begin{aligned}
(\hat{T}_{g_a}(\hat{U}_{(ab)}\Psi))[U] &= (\hat{L}_{g_a}^{(ab)}(\hat{U}_{(ab)}\Psi))[\{U_{(ab)}\}] && \text{from Eq. (32)} \\
&= (\hat{U}_{(ab)}\Psi)[\{g_aU_{(ab)}\}] && \text{from Eq. (29)} \\
&= g_a^rU_{(ab)}^r\Psi[\{g_aU_{(ab)}\}] && \text{from Eq. (37)} \\
&= g_a^rU_{(ab)}^r\Psi[\{U_{(ab)}\}] && \text{from gauge invariance of } \Psi \\
&\neq (\hat{U}_{(ab)}\Psi))[U] = U_{(ab)}^r\Psi[\{U_{(ab)}\}],
\end{aligned} \tag{38}$$

so the functional $\hat{U}_l\Psi$ is not necessarily in $\mathcal{H}_{ginv}$ for an arbitrary $\Psi \in \mathcal{H}_{ginv}$. This is what we call a non-gauge-invariant operator.

The gauge-invariant observables composed using $\hat{U}_l$ are **Wilson loops**, i.e.

$$\hat{W}_\Gamma = \prod_{l\in\Gamma}\hat{U}_l, \tag{39}$$

where $\Gamma = ((a_1, a_2), (a_2, a_3), ..., (a_{n-1}, a_n), (a_n, a_1))$ is an oriented closed path consisting of links in the lattice. To see that, consider a triangle of nodes a, b, c with links $(a, b), (b, c), (c, a)$:

$$\begin{aligned}
(\hat{T}_{g_a}(\hat{U}_{(ab)}\hat{U}_{(bc)}\hat{U}_{(ca)}\Psi))[U] &= (\hat{L}_{g_a}^{(ab)}\hat{L}_{g_a}^{(ac)}(\hat{U}_{(ab)}\hat{U}_{(bc)}\hat{U}_{(ca)}\Psi))[\{U_{(ab)}, U_{(bc)}, U_{(ca)}\}] && \text{from Eq. (32)} \\
&= (\hat{L}_{g_a}^{(ab)}\hat{L}_{g_a^{-1}}^{(ca)}(\hat{U}_{(ab)}\hat{U}_{(bc)}\hat{U}_{(ca)}\Psi))[\{U_{(ab)}, U_{(bc)}, U_{(ca)}\}] && \text{from Eq. (30)} \\
&= (\hat{U}_{(ab)}\hat{U}_{(bc)}\hat{U}_{(ca)}\Psi))[\{g_aU_{(ab)}, U_{(bc)}, g_a^{-1}U_{(ca)}\}] && \text{from Eq. (29)} \\
&= g_a^rU_{(ab)}^rU_{(bc)}^r(g_a^{-1})^rU_{(ca)}^r\Psi[\{g_aU_{(ab)}, U_{(bc)}, g_a^{-1}U_{(ca)}\}] && \text{from Eq. (37)} \\
&= U_{(ab)}^rU_{(bc)}^rU_{(ca)}^r\Psi[\{g_aU_{(ab)}, U_{(bc)}, g_a^{-1}U_{(ca)}\}] && \text{as } U_l^r \text{ are scalars,} \\
&= U_{(ab)}^rU_{(bc)}^rU_{(ca)}^r\Psi[\{U_{(ab)}, U_{(bc)}, U_{(ca)}\}] && \text{from gauge invariance of } \Psi \\
&= (\hat{U}_{(ab)}\hat{U}_{(bc)}\hat{U}_{(ca)}\Psi))[U].
\end{aligned} \tag{40}$$

As the gauge transformation $\hat{T}_{g_a}$ was at an arbitrary node a, the same will hold for any $\hat{T}_{g_n}$ at an arbitrary node $n \in \Gamma$. One can repeat this proof for a loop of an arbitrary length.

Together, the link operators $\hat{L}_g^l$ and the Wilson loops $\hat{W}_\Gamma$ generate the entire set of possible gauge-invariant operators [8]. With this setup, we can proceed to calculating entanglement entropy.

4 Algebra of observables approach

4.1 How to define a quantum state on an algebra?

This section follows the approach of Casini et al. [8].

In the algebraic approach to QFT, a state can be defined directly on an algebra of observables on a vector space, without reference to a Hilbert space of states.

A **state on an algebra** is defined as [23]:

$$\omega : \mathcal{B}(\mathcal{V}) \to \mathbb{C} \tag{41}$$

with the following properties:

1. ω is linear
2. for all positive definite $B \in \mathcal{B}(\mathcal{V})$, $\omega(B \in \mathcal{B}(\mathcal{V})) > 0$
3. $\omega(\mathbb{1}) = 1$.

This is not unique - given an algebra of observables, there may be multiple definitions of ρ which meet the aforementioned criteria. Here, we will work with the choice:

$$\omega(B) = \mathrm{Tr}(\rho B), \tag{42}$$

where given a state $|\Psi\rangle \in \mathcal{H}$, the corresponding $\rho \in \mathcal{B}(\mathcal{H})$ is the unique operator reproducing the expectation values of all observables from the usual definition of a state:

$$\mathrm{Tr}(\rho B) = \langle\Psi| B |\Psi\rangle \quad \forall \Psi \in \mathcal{H}. \tag{43}$$

Crucially, the algebra of observables considered here admits a trace, and the unique ρ exists. In fact, this ρ is precisely the density matrix. So far, there has been nothing particularly new - the discussion above is just a different formalisation of the density matrix.

4.2 How to calculate the entanglement entropy given a state on an algebra?

To calculate the entanglement entropy, we need **local density matrices**. Here is where this approach shows its advantages. By construction, the operators $\hat{L}^l_g$ and $\hat{U}_l$ are associated to a link, and hence local. Given a set A of links, the algebra generated by all operators on links within A is denoted $\mathcal{A}$.

The local density matrix ρ_A is defined to be an operator $\rho_A \in \mathcal{B}(\mathcal{H})$ such that:

$$\mathrm{Tr}(O_A \rho_A) = \langle\Psi| O_A |\Psi\rangle \quad \forall \Psi \in \mathcal{H}, \quad O_A \in \mathcal{A}. \tag{44}$$

If such an operator can be found, the entanglement entropy can then be simply calculated as:

$$S_A = -\mathrm{Tr}_{\mathcal{A}} \rho_A \log \rho_A. \tag{45}$$

Note that ρ_A is not constructed by partial tracing over the outside region $\bar{A}$, so there is no need for the undelying Hilbert space to decompose into tensor products. The trace $\mathrm{Tr}_{\mathcal{A}}$ is the trace on $\mathcal{A}$, again without reference to any Hilbert space. Further simplifications are possible by identifying that all operators in question are block-diagonal, which will be explored below.

4.3 What is a superselection sector?

In gauge theories, one can construct Gauss-like quantities which, informally speaking, share information between inside and outside regions. An example of this is the electric flux in electromagnetism - the charge inside a region can be calculated using the value of flux flowing into the outside. This is why gauge-invariant Hilbert spaces cannot be decomposed into local tensor product spaces.

In the algebra of observables language, gauge invariance leads to the existence of **superselection sectors**, i.e. sets of states corresponding to the same value of all possible Gauss-like constraints. If two different states lie in the same superselection sector, one can find a gauge-invariant operator B such that $B|\Psi\rangle = |\Psi'\rangle$. Conversely, if they lie in two different superselection sectors, such a gauge-invariant operator B does not exist. This enables us to decompose the operators in $\mathcal{A}$ into a block-diagonal form, which in turn leads to a decomposition of entanglement entropy.

4.4 Centre of the gauge-invariant algebra

Consider a rectangular region A of links on the square lattice and its gauge-invariant algebra $\mathcal{A}$. Any operator $O_A \in \mathcal{A}$ will commute with all operators in $\bar{\mathcal{A}}$ - this can be checked directly from the definition. Therefore, $\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)}$ commutes with all $O_{\bar{A}} \in \bar{\mathcal{A}}$. However, $L_g^{(51)} = \hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)}$ from gauge invariance as written in Eq. (34), so $\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)} \in \bar{\mathcal{A}}$. Hence, $\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)}$ commutes with all $O_A \in \mathcal{A}$.

Therefore, $\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)}$ commutes with all operators in the gauge-invariant algebra, $\mathcal{A} \cup \bar{\mathcal{A}}$. Denoting $\mathcal{A}'$ as the commutant of $\mathcal{A}$, we get:

$$\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)} \in \mathcal{A}' \cap \bar{\mathcal{A}}' = (\mathcal{A} \cup \bar{\mathcal{A}})' \equiv \mathcal{Z}, \tag{46}$$

i.e. the operator belongs to the **centre of the gauge-invariant algebra**, $\mathcal{Z}$. The Gauss-like constraints are precisely the eigenvalues of these operators. In the Abelian case, in a given superselection sector every state in $\mathcal{H}_{ginv}$ belonging to that sector will have the same eigenvalues of the generators of the centre $\{\mathcal{O}_i\}$, denoted $(\lambda_1, ..., \lambda_n)$.

4.5 Decomposing the entanglement entropy

The generators $\mathcal{O}_i$ can be simultaneously diagonalised to get:

$$\begin{pmatrix} \lambda^1\mathbb{1} & 0 & \cdots & 0 \\ 0 & \lambda^2\mathbb{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^m\mathbb{1} \end{pmatrix}, \tag{47}$$

where each $\mathbb{1}$ has some dimension $d_k \times d_k$. As both $\mathcal{A}$ and $\mathcal{A}'$ commute with the centre, they both have block diagonal forms. Hence, every element of the algebra generated by operators in $\mathcal{A}$ and $\mathcal{A}'$ can be written as a block diagonal product of matrices:

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix} \times \begin{pmatrix} A'_1 & 0 & \cdots & 0 \\ 0 & A'_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A'_n \end{pmatrix} = \begin{pmatrix} A_1A'_1 & 0 & \cdots & 0 \\ 0 & A_2A'_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_nA'_n \end{pmatrix} \tag{48}$$

Hence, each of the block-diagonal sectors is in fact decomposable into a tensor product of algebras, which means we can take the partial trace over $\mathcal{A}_i'$ within individual sectors to obtain the reduced density matrix:

$$\rho_A = \begin{pmatrix} p_1\rho_{\mathcal{A}_1} & 0 & \cdots & 0 \\ 0 & p_2\rho_{\mathcal{A}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n\rho_{\mathcal{A}_n} \end{pmatrix}, \tag{49}$$

where the probabilities p_i were added to renormalise the blocks so that $\mathrm{Tr}\rho_{\mathcal{A}_i} = 1$.

The entanglement entropy of a block-diagonal matrix (or equivalently, a direct sum of operators) can be then decomposed as:

$$S_A = -\mathrm{Tr}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}) = -\sum_i p_i \log(p_i) + \sum_i p_i S(\rho_{\mathcal{A}_i}) \equiv H(\{p_i\}) + \sum_i p_i S(\rho_{\mathcal{A}_i}) \tag{50}$$

The first term is the classical Shannon entropy of the probability distribution of the sectors. The second term is the quantum entanglement entropy within each sector.

This will be useful in calculations - we can first *fix* the superselection sector, calculate the quantum part of the entropy for that specific sector, repeat for each sector or use symmetry properties, and separately - compute the probability distribution of the sectors. In this way, we can get the full entanglement entropy.

4.6 Algebra of observables: $\mathbb{Z}_2$ theory

4.6.1 Hamiltonian and ground state definition

A useful example to calculate is is what happens in $\mathbb{Z}_2$ theory, for the widely studied Kogut-Susskind Hamiltonian [17]:

$$\hat{H} = \frac{1}{2\beta_g} \sum_l \hat{\sigma}_l - \beta_g \sum_p \prod_{l \in p} \hat{z}_l. \tag{51}$$

where β_g is the coupling constant, $\sum_l$ and $\sum_p$ denote summation over all lattice links and plaquettes and $\hat{z}_l$ and $\hat{\sigma}_l$ are two anticommuting operators, both with eigenvalues ± 1.
The trivial ground state, $\Psi_0 = \text{constant}$, can be excited with the Wilson loop operators. Consider exciting the ground state with all possible loops:

$$\Psi[U] = N \sum_\Gamma \prod_{l \in \Gamma} U_l, \tag{52}$$

where the sum is over all possible loops Γ on the lattice, N is a normalisation constant, and the variables U_l are just the values of the gauge fields at each link, $U_l \in \{-1, 1\}$. This state can be rewritten as:

$$\Psi[U] = N \sum_\Gamma \hat{W}_\Gamma \Psi_0, \tag{53}$$

$$\tag{54}$$

Both the vacuum, Ψ_0, and all the Wilson loop operators W_Γ are gauge-invariant, so $\Psi[U]$ is gauge-invariant.

4.6.2 Invariance of $|\Psi\rangle$ under W_Γ

What happens if we act with a single Wilson loop operator W_Γ on $\Psi[U]$? Note that in $\mathbb{Z}_2$ theory $W_\Gamma^2 = \mathbb{1}$, i.e. double action with the same Wilson loop operator flips each link twice, so it returns the original configuration. Hence $W_\Gamma^{-1} = W_\Gamma$. Therefore, for every W_{Γ_i} already present in the superposition $\Psi[U]$, there exists a $W_{\Gamma_i'} = W_\Gamma W_{\Gamma_i}$, such that $W_\Gamma W_{\Gamma_i'} = W_{\Gamma_i}$. This assignment is unique, so multiplying the set of all W_{Γ_i} by W_Γ generates the set of all W_{Γ_i} again. Therefore,

$$\Psi[U] = W_\Gamma \Psi[U] \quad \forall \text{ closed loops } \Gamma. \tag{55}$$

One can commute W_Γ with $\hat{L^l_{-1}}$ if $l \notin \Gamma$, and anticommute W_Γ with $\hat{L^l_{-1}}$ if $l \in \Gamma$. To see this, note that:

1. If $l \notin \Gamma$, flipping an edge outside the loop does not change $\hat{U}_l \Psi$ for any $l \in \Gamma$, so they commute. Hence $W_\Gamma \hat{L^l_{-1}} = \hat{L^l_{-1}} W_\Gamma$.

2. If $l \in \Gamma$, flipping exactly 1 link inside the loop causes the product of all $l \in \Gamma$ to flip. Hence $W_\Gamma \hat{L^l_{-1}} = -\hat{L^l_{-1}} W_\Gamma$.

4.6.3 Why do constraints fully describe the quantum state?

The maximal algebra of observables is the set of all polynomials of W_Γ and $\hat{L^l_g}$. Denote $\hat{L^l_1} = \mathbb{1}$, $\hat{L^l_{-1}} = \sigma_x^l$. Every such polynomial can be written as a linear combination of operators in the form of:

$$\pm \prod_{l=1}^{N} \sigma_x^l W_\Gamma \tag{56}$$

To see this, note that:

1. The identity operators do not contribute.

2. The Wilson loops inside any polynomial can be commuted or anticommuted to the right.

3. The product of all Wilson loops can be replaced by a single loop.

Also note that:

$$\langle \Psi | \prod_{i=1}^{N} \sigma_x^{l_i} | \Psi \rangle = 0 \tag{57}$$

if there exists a Wilson loop passing through an odd number of links $l \in \{l_i\}$. To see this, note:

$$\langle \Psi | \prod_{i=1}^{N} \sigma_x^{l_i} | \Psi \rangle = \langle \Psi | \prod_{i=1}^{N} \sigma_x^{l_i} W_\Gamma | \Psi \rangle \quad \text{by Eq. (55)} \tag{58}$$

$$= - \langle \Psi | W_\Gamma \prod_{i=1}^{N} \sigma_x^{l_i} | \Psi \rangle \text{ by commuting } W_\Gamma \tag{59}$$

$$= 0. \tag{60}$$

Following the vocabulary of Casini et al., we will call a product $\prod_{i=1}^{N} \sigma_x^{l_i}$ such that there **does not** exist a Wilson loop passing through an odd number of $\sigma_x^{l_i}$ within $\prod_{i=1}^{N} \sigma_x^{l_i}$ a *constraint*. An equation of form $\prod_{i=1}^{N} \sigma_x^{l_i} = \mathbb{1}$ is a *constraint equation*.

Now suppose that $\mathcal{O}_1$, $\mathcal{O}_2$ are linear combinations of $\prod_{i=1}^{N} \sigma_x^{l_i}$ such that it is not possible to create a new constraint using minimum 1 constraint from $\mathcal{O}_1$ and 1 constraint from $\mathcal{O}_2$. Then, the expectation value of a product decomposes into individual operators:

$$\langle \Psi | \mathcal{O}_1 \mathcal{O}_2 | \Psi \rangle = \langle \Psi | \mathcal{O}_1 | \Psi \rangle \langle \Psi | \mathcal{O}_2 | \Psi \rangle \tag{61}$$

as all the products $\prod_{i=1}^{N} \sigma_x^{l_i}$ within $\mathcal{O}_1 \mathcal{O}_2$ which are not either in $\mathcal{O}_1$ or $\mathcal{O}_2$ separately will not be constraints and thus will have expectation value 0. This is important, as it means that the only correlations within the systems are between constraint operators. To find the right state ρ on the gauge-invariant algebra, we just need to find a state that reproduces the right expectation values for the maximal set fulfilling the condition imposed here on $\mathcal{O}_1, \mathcal{O}_2$, as it will automatically reproduce all other expectation values.

Is there a way to identify all operators generating the constraint equations? As it turns out, the following holds true:

$$\mathcal{O}_T = \prod_{i=1}^{N} \hat{T}^{g_i, a_i} \implies \mathcal{O}_T \text{ is a constraint,} \tag{62}$$

i.e. products of local gauge transformations are constraints.

To see this, we will show that there exist no Wilson loops on the lattice which pass through an odd number of the links in $\mathcal{O}_T$. Recall that local gauge transformations act on all links pointing outwards from a chosen vertex. In a square grid geometry, this gives them a shape of a 'cross'. We will reason graphically, with red dots denoting the vertex of gauge transformation, and blue lines - all σ_x^l present in the gauge transformation operator.

First notice that multiplying the 'crosses' on 2 neighbouring vertices will result in cancellation of the σ_x^l on the link between them as the operators on it point in different directions - always outward from the red point. The red line on which the blue operators cancel out is called a *constraint line*.

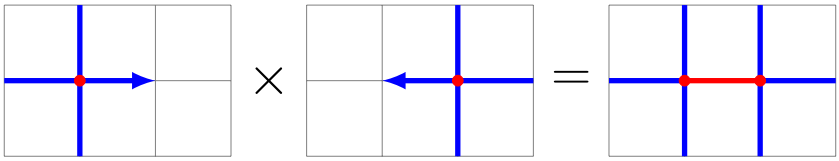

Figure 3: Graphical demonstration of the operator $\hat{T}_{\sigma_x, v=1} \hat{T}_{\sigma_x, v=2}$, i.e. a gauge transformation by σ_x at vertices $1, 2$.

Extending this to products of multiple crosses, the final operator resulting from this multiplication is a product of links (in blue) which are all links attached to the constraint line, but not *on* the constraint line (in red). Figure 4 contains 3 examples of such an operator: 1 simple cross, 1 with an open constraint line, and 1 with a closed constrained line (a red loop).

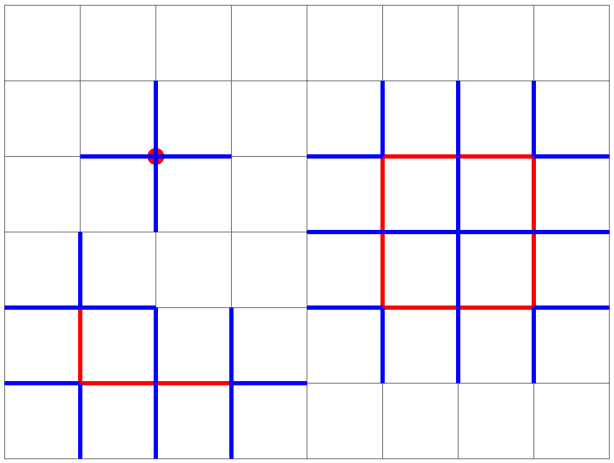

Figure 4: Examples of constraint lines (in red) and their associated sets of attached links (in blue). The sets of blue links are the operators $\mathcal{O}_T = \prod_{i=1}^{N} \hat{T}^{\sigma_x, a_i}$, or equivalently $\prod_l \sigma_x^l$ where l are the blue links.

Now we are ready to answer the question: can a Wilson loop pass through an odd number of blue links? The answer lies in the following observations:

- If a loop enters a vertex on a constraint line, it must do so through a single blue link.
- If a loop moves along a constraint line, it does not cross any blue links.
- If a loop exits a vertex on a constraint line, it must do so through a single blue link.
- If a loop enters a constraint line, it must exit it.

Combining the arguments above, we see that if a loop crosses a constraint line, it must pass an even number of blue links.

What if a loop passes through blue links, but not red? This is indeed possible, but only on corners and ends of open lines. Hence, a loop will always pass through 2 perpendicular blue links consecutively. The simplest examples is a loop passing through the red dot on the image. Therefore, any Wilson loop must pass a set of blue links as defined above an even number of times.

In the case of a closed loop, this picture can be further simplified by choosing only the 'outside' blue links to form $\hat{T}_{\partial A}$. For the same reasons as above, all Wilson loops must pass through an even number of blue links in the operator.

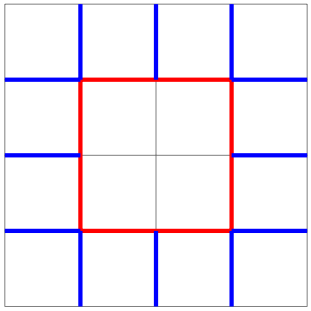

Figure 5: Constraint operator $\hat{T}_{\partial A}$ at the boundary of an enclosed region.

The key point is: in order to find ρ, we must find an operator which reproduces the expectation values $\langle\Psi|\,\mathcal{G}_i\,|\Psi\rangle$ for all independent blue links, or products thereof. With this knowledge, we can finally try to find ρ and calculate its entanglement entropy.

4.6.4 Classical entropy from the probability distribution of superselection sectors

Consider a rectangular region A, for which we want to calculate the entanglement entropy. Take the set $\{\mathcal{G}_i\}$ to be all operators σ_x^l with 1 vertex on the boundary, and $\sigma_x^l\sigma_x^k$ for the ones attached to corners (see Figure 6). Then, there is exactly 1 independent constraint which can be formed from all of the operators, namely $\prod_{i=1}^{N}\mathcal{G}_i$. After discarding any one of these operators, no constraint can be formed using $\mathcal{G}_i$, so the expectation value always vanishes:

$$\forall_i\,\langle\Psi|\,\mathcal{G}_i\,|\Psi\rangle = 0 \tag{63}$$

$$\forall_{i,j}\,\langle\Psi|\,\mathcal{G}_i\mathcal{G}_j\,|\Psi\rangle = 0,\ldots \tag{64}$$

or more concisely,

$$\forall\{G_{i_j}\}\,\langle\Psi|\prod_{j=1}^{M\leq N}\mathcal{G}_{i_j}\,|\Psi\rangle = 0, \tag{65}$$

i.e. any product of 1 or more generators has expectation value 0. As $\mathcal{G}_i$ is a complete set of N generators of the center, the gauge-invariant Hilbert space is divided into 2^N superselection sectors corresponding to all possible sets of eigenvalues of respective generators, $(\lambda_1,\lambda_2,\ldots,\lambda_N)$. The density matrix can then be expressed as a weighted sum of density matrices corresponding to the superselection regions:

$$\rho = \sum_{\{\vec{\lambda}\}} p_\lambda \rho_\lambda. \tag{66}$$

Claim

The condition Eq. (65) implies that the probability of each superselection sector, p_λ must be the same, and thus equal to 2^{-N}. This complements the discussion in Section 5 of [8], where this result was cited.

Proof

From the definition of the sectors and ρ_λ, the value of $\mathrm{Tr}\left(\rho_\lambda \mathcal{G}_i\right)$ is just λ_i, and similarly for any product of generators:

$$\langle\Psi| \prod_{j=1}^{M\leq N} \mathcal{G}_{i_j} |\Psi\rangle = \sum_{\{\vec{\lambda}\}} p_\lambda \left(\rho_\lambda \prod_{j=1}^{M\leq N} \mathcal{G}_{i_j}\right) \tag{67}$$

$$= \sum_{\{\vec{\lambda}\}} p_\lambda \prod_{j=1}^{M\leq N} \lambda_{j_i} \tag{68}$$

for any subset of generators $\{\mathcal{G}_{j_i}\}$ For example, if there are 3 generators, and $\vec{\lambda} = (-1, 1, 1)$, then $\rho_\lambda \mathcal{G}_0\mathcal{G}_1 = -1 \times 1 = -1$. Using the isomorphism of $(1, -1, \times) \simeq (0, 1, \oplus)$, where $\oplus$ denotes addition modulo 2, and writing β_i as the binary counterpart of λ_i, i.e. $\lambda_i = 1 \longleftrightarrow \beta_i = 0$, $\lambda_i = -1 \longleftrightarrow \beta_i = 1$, Eq. (65) can be rewritten as:

$$\langle\Psi| \prod_{j=1}^{M\leq N} \mathcal{G}_{i_j} |\Psi\rangle = \sum_{\{\vec{\lambda}\}} p_\lambda \left(-1\right)^{\sum_{j=1}^{M} \beta_{i_j}} = 0. \tag{69}$$

Consider a system of such equations generated by the power set of $\{\mathcal{G}_i\}$, i.e. all unique products of generators where each generator appears maximum once. First, consider an example with 2 generators, $\mathcal{G}_0$ and $\mathcal{G}_1$. The system of equations for probabilities is:

$$\begin{aligned}
\mathrm{Tr}\left(\rho\right) = 1 &\Longleftrightarrow p_{00} + p_{01} + p_{10} + p_{11} = 1 \\
\mathrm{Tr}\left(\rho\, \mathcal{G}_0\right) = 0 &\Longleftrightarrow p_{00} - p_{01} + p_{10} - p_{11} = 0 \\
\mathrm{Tr}\left(\rho\, \mathcal{G}_1\right) = 0 &\Longleftrightarrow p_{00} + p_{01} - p_{10} - p_{11} = 0 \\
\mathrm{Tr}\left(\rho\, \mathcal{G}_0\mathcal{G}_1\right) = 0 &\Longleftrightarrow p_{00} - p_{01} - p_{10} + p_{11} = 0,
\end{aligned} \tag{70}$$

where the first condition comes from the fact that ρ is a density matrix, and e.g. p_{00} denotes the probability of sector $\boldsymbol{\beta} = (0, 0)$. The only solution to this system of equations is $p_{00} = p_{01} = p_{10} = p_{11} = \frac{1}{4}$.

Now, we will generalise this to any N of generators, yielding 2^N equations. Eq. (70) can be written in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{71}$$

For future reference, denote:

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{72}$$

Suppose we want to add 1 more generator to the set. Then, the new equations introduced will correspond to each of the old equations with the product of generators multiplied by $\mathcal{G}_2$, e.g. $\mathrm{Tr}\left(\rho\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_0\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_1\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_0\mathcal{G}_1\right) \rightarrow \mathrm{Tr}\left(\rho\, \mathcal{G}_2\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_0\, \mathcal{G}_2\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_1\, \mathcal{G}_2\right), \mathrm{Tr}\left(\rho\, \mathcal{G}_0\mathcal{G}_1\, \mathcal{G}_2\right)$. Let H_{2^N} be the matrix generating the system of equations for N generators. Then $H_{2^{N+1}}$ can be constructed

from it. First, notice that the row corresponding to $\mathrm{Tr}\,(\rho\,\mathcal{G}_{N+1})$ will always look like:

$$\left(\underbrace{1,\ldots,1}_{2^N},\underbrace{-1,\ldots,-1}_{2^N}\right) \tag{73}$$

as it only depends on the value of $(-1)^{\beta_{N+1}}$, which in increasing order of the corresponding binary numbers is 0 in the first 2^N numbers and 1 in the second 2^N numbers. The rows without $\mathcal{G}_{N+1}$ are independent of $(-1)^{\beta_{N+1}}$, and hence will be equal to the matrix H_{2^N} repeated twice side-by-side. Hence, the first 2^N+1 rows will look like:

$$\begin{pmatrix} H_{2^N} & H_{2^N} \\ (1) & (-1) \end{pmatrix}. \tag{74}$$

Then, the rest of the rows are products of rows within H_{2^N} and β^{N+1}. Hence, the final matrix is:

$$\begin{pmatrix} H_{2^N} & H_{2^N} \\ H_{2^N} & -H_{2^N} \end{pmatrix}. \tag{75}$$

Eq. (75) together with the initial H_4 matrix in Eq. (72) turns out to be the Sylvester construction for Hadamard matrices [14]. Hadamard matrices have the following property:

$$H_{2^N} H_{2^N}^\top = 2^N \mathbb{1}. \tag{76}$$

Hence, the probabilities will obey:

$$H_{2^N} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{2^N-1} \\ p_{2^N} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{77}$$

Multiplying by $H_{2^N}^\top$ yields:

$$2^N \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{2^N-1} \\ p_{2^N} \end{pmatrix} = H_{2_N}^\top \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{78}$$

Finally, it follows that the probabilities must be all equal:

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{2^N-1} \\ p_{2^N} \end{pmatrix} = \begin{pmatrix} 2^{-N} \\ 2^{-N} \\ \vdots \\ 2^{-N} \\ 2^{-N} \end{pmatrix}. \tag{79}$$

The classical entropy associated with the probability distribution of superselection sectors is:

$$H = -\sum_{\{\boldsymbol{\lambda}\}} p_\lambda \log p_\lambda = -2^N \times 2^{-N} \log 2^{-N} = N \log 2, \tag{80}$$

where N is the number of independent generators of the centre. □

How many independent generators are there given a single bounded region? All links coming out of a single boundary vertex form a single independent generator. Remember that we discarded 1 generator from the set so that no constraint equations can be formed using $\mathcal{G}_i$. Similarly, each disconnected boundary will require discarding 1 generator. Therefore, the number of independent generators is

$$N = N_{\text{boundary nodes}} - n_\partial = L - n_\partial, \tag{81}$$

where n_∂ is the number of disconnected boundaries, and L - the number of links on the boundary. Thus, the classical entanglement entropy is:

$$S_A = (L - n_\partial) \log 2 \tag{82}$$

The generators in the electric centre are shown on Figure 6.

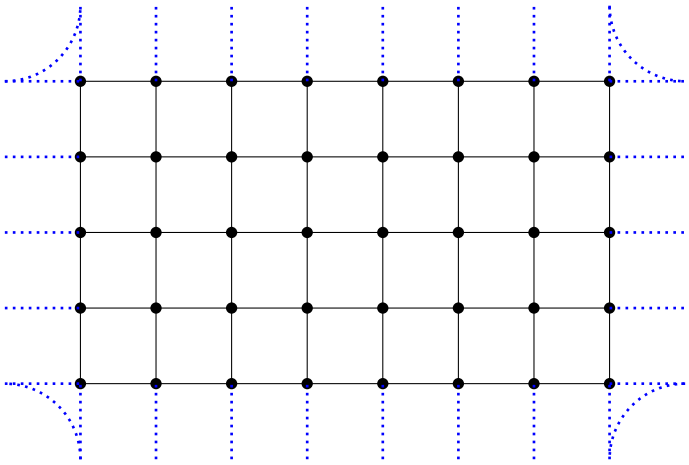

Figure 6: Generators of the electric centre $\mathcal{Z}$, highlighted in blue. The pairs attached to each corner are a single generator.

4.6.5 Quantum entropy within a single superselection sector

Now we will calculate the quantum entropy within a specific superselection sector. Choose any particular $\boldsymbol{\lambda}$. Let $|\Psi_\lambda\rangle$ be a simultaneous eigenstate of generators $\mathcal{G}_i$ with eigenvalues $\boldsymbol{\lambda}$. The algebra $\mathcal{A}_\lambda$ of all operators on the Hilbert space restricted to all the states of form $|\Psi_\lambda\rangle$, i.e. $\mathcal{H}_{\boldsymbol{\lambda}}$, is then formed by all the projections:

$$O_V^\lambda = P_{\boldsymbol{\lambda}} O_V P_{\boldsymbol{\lambda}}, \tag{83}$$

where

$$P_{\boldsymbol{\lambda}} = \prod_{i=1}^{N} \frac{1}{2}(1 + \lambda_i \mathcal{G}_i). \tag{84}$$

Notice that $P_{\boldsymbol{\lambda}}$ commute with all other operators. The complete set of commuting operators within a fixed $\boldsymbol{\lambda}$ is given only by the Wilson loop operators, W_Γ. As W_Γ is Hermitian, their expectation value is:

$$\langle\Psi| P_{\boldsymbol{\lambda}} W_\Gamma P_{\boldsymbol{\lambda}} |\Psi\rangle = \langle\Psi| W_\Gamma P_{\boldsymbol{\lambda}}^2 |\Psi\rangle = \langle\Psi| W_\Gamma P_{\boldsymbol{\lambda}} |\Psi\rangle = \langle\Psi| P_{\boldsymbol{\lambda}} |\Psi\rangle = 2^{-N}. \tag{85}$$

From the definition of the reduced density matrix we get:

$$p_{\boldsymbol{\lambda}} \mathrm{Tr}(\rho_V^{\lambda} W_\Gamma) = \langle \Psi | W_\Gamma^{\lambda} | \Psi \rangle, \tag{86}$$

i.e.

$$2^{-N} \mathrm{Tr}(\rho_V^{\lambda} W_\Gamma) = 2^{-N} \iff \mathrm{Tr}(\rho_V^{\lambda} W_\Gamma) = 1. \tag{87}$$

Since the trace is the sum of the eigenvalues weighted by their probabilities in the state, the state must have probability 1 of being found in an eigenstate of W_Γ with an eigenvalue +1, and probability 0 of being found in an eigenstate of W_Γ with an eigenvalue -1. Hence, if ρ_V^{λ} is a mixed state, it must be a mixture only of states with eigenvalue +1 for every W_Γ. As W_Γ generate the entire algebra in the chosen superselection sector, there are no 2 distinct eigenstates with eigenvalue +1 for all W_Γ. This means ρ_V^{λ} must be a single eigenstate of all W_Γ, i.e. a pure density matrix. Therefore, the quantum entropy of ρ_V^{λ} is 0.

This is consistent with [30], where it has been proven that the entanglement entropy of pure Abelian lattice gauge theories with no matter will always come only from the classical Shannon entropy term, $H(\{p_i\})$.

Finally, we obtain the full entanglement entropy:

$$S_A = H_A + \sum_{\{\boldsymbol{\lambda}\}} S_{\rho^{\boldsymbol{\lambda}}} = (L - n_\partial) \log 2 + 0 = S_A = (L - n_\partial) \log 2. \tag{88}$$

This method can be extended to other Abelian gauge theories - a calculation for $U(1)$ has been pursued in [7].

4.7 Ambiguity in the choice of the centre

In the calculation above, we have assumed that the links on the boundary are still inside the region. However, if we define a *region* of space as a set of lattice sites, one can equivalently remove any number of those links from the inside region. This would yield a different set of inside links, and thus a different algebra of local gauge-invariant observables. For example, the opposite choice is that every boundary link is outside. This is often called the magnetic centre.

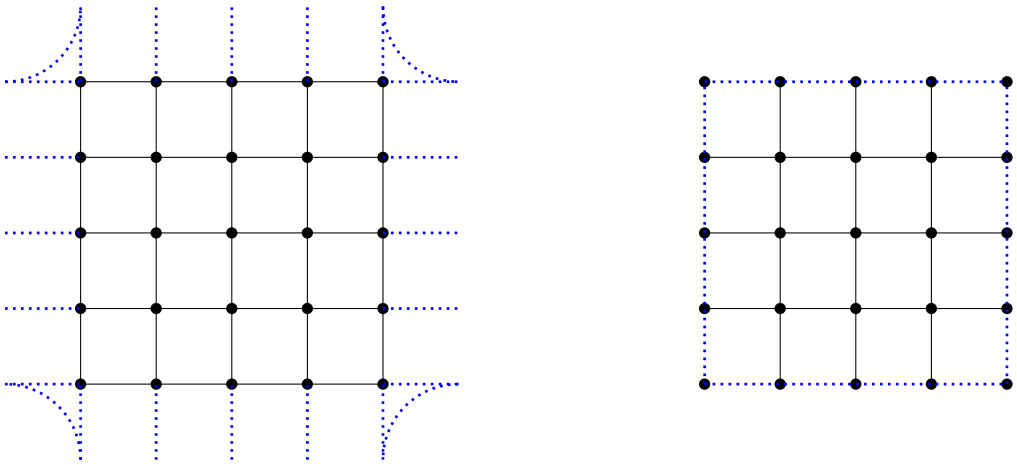

Figure 7: Generators of the electric centre (left) and the magnetic centre (right), highlighted in blue. The pairs attached to each corner in the electric centre are a single generator. The boundary Wilson loop in the magnetic centre is a single generator.

In the magnetic centre case, there is only 1 operator in the centre - the Wilson loop on the boundary, W_∂. This must be equal to the identity operator, whose only eigenvalue is 1. Hence, the expectation value of W_∂ on the gauge-invariant state must be 1, and thus the only superselection sector present is one for which $\lambda_1 = 1$. From this, the classical entropy vanishes. Within one sector, we can repeat the reasoning for the electric centre above as the expectation values for all other loops are unchanged. Thus, the quantum entropy also vanishes.

This is striking - we got a different value of entanglement entropy based on the choice of link inclusion, and thus choice of centre. In general, the algebra of observables method requires a centre prescription, or else it may give different results for different centers. For later comparison with other methods, we will use the electric centre.

5 Extended Hilbert space (GST) approach

5.1 Key idea: extended Hilbert space by separating all lattice links

This section follows the approach of Ghosh, Soni and Trivedi [13, 27, 28].

Instead of calculating the entanglement entropy of the local density operator ρ_A, we can embed the gauge-invariant state $|\Psi\rangle$ into a larger Hilbert space $\mathcal{H}$, which can be decomposed into a tensor product of local Hilbert spaces. This is a huge advantage of this method - the entanglement entropy can then be calculated using the original definition, and there is no need to find the appropriate ρ, which arguably may be difficult.

The problem is that $\mathcal{H}$ will not be gauge-invariant anymore, and the embedding is not unique. When choosing $\mathcal{H}$, one has to then justify: why are we choosing this particular extended Hilbert space?

Ghosh, Soni and Trivedi [13] have opted for an extended $\mathcal{H}$ which is the tensor product of Hilbert spaces living on individual links. This in a sense 'maximal' approach maintains the lattice structure, and provides a natural tensor product basis by construction.

5.2 Basis states and observables

The extended Hilbert space H is simply:

$$\bigotimes_{l \in V} \mathcal{H}_l, \tag{89}$$

where V is the set of all links.

The entanglement entropy can be calculated simply as:

$$S_{in} = -\mathrm{Tr}(\rho_{in} \log \rho_{in}), \tag{90}$$

where ρ_{in} is the density matrix of the gauge invariant state in Eq. (52) considered as a member of the extended Hilbert space Eq. (89).

For $\mathbb{Z}_2$ theory, we can take the orthonormal basis on each link to be the eigenstates of σ_x, i.e.

$$\sigma_x^l \left|\pm 1\right\rangle_l = \pm \left|\pm 1\right\rangle_l . \tag{91}$$

Define a Hilbert space which is a tensor product of all outside links with 1 vertex at the boundary:

$$\mathcal{H}_{\partial V,out} = \bigotimes_{l\in\partial V} \mathcal{H}_l. \tag{92}$$

Then, choose a basis for $\mathcal{H}_{\partial V,out}$ to be the tensor product of the bases from Eq. (91) for every link on the boundary:

$$|\psi\rangle_{\partial V} = \bigotimes_{l\in\partial V} |\pm 1\rangle_l \,. \tag{93}$$

Similarly, define the Hilbert space $\mathcal{H}_{out}$ as

$$\mathcal{H}_{out} = \bigotimes_{l\in\bar{V}} \mathcal{H}_l. \tag{94}$$

The Hilbert space of links inside the region V is

$$\mathcal{H}_{in} = \bigotimes_{l\in V} \mathcal{H}_l. \tag{95}$$

5.3 Expressing the states in terms of superselection sectors

As before, define a vector, $\boldsymbol{\lambda}$, whose entries give the eigenvalues of the total *electric flux operators*, i.e. σ_x^l going outside at the boundary vertices. These are exactly the generators $\mathcal{G}_i$ from Section 4.6.4. As each basis state in $\mathcal{H}_{out}$ has a well-defined value of outgoing electric flux at each boundary vertex, $\mathcal{H}_{out}$ can be written as a direct sum:

$$\mathcal{H}_{out} = \bigoplus_{\{\boldsymbol{\lambda}\}} \mathcal{H}_{out}^{\boldsymbol{\lambda}} \tag{96}$$

Analogously, each basis state in $\mathcal{H}_{in}$ has a well-defined value of ingoing electric flux at each boundary vertex, thus:

$$\mathcal{H}_{in} = \bigoplus_{\{\boldsymbol{\lambda}\}} \mathcal{H}_{in}^{\boldsymbol{\lambda}}. \tag{97}$$

How to express ρ in terms of states within each superselection sector? From Eq. (96) and Eq. (97), it follows that the projection operators onto each sector form a complete resolution of identity:

$$\mathbb{1}_{\mathcal{H}_{out}} = \sum_{\{\boldsymbol{\lambda}\}} \hat{P}^{\boldsymbol{\lambda}}_{\mathcal{H}_{out}} \,; \tag{98}$$

$$\mathbb{1}_{\mathcal{H}_{in}} = \sum_{\{\boldsymbol{\lambda}\}} \hat{P}^{\boldsymbol{\lambda}}_{\mathcal{H}_{in}}. \tag{99}$$

Therefore, any state $|\psi\rangle \in \mathcal{H}_{in}$ can be expressed as:

$$\rho_{in} = \mathrm{Tr}_{\mathcal{H}_{out}} |\psi\rangle\langle\psi| = \sum_{\{\boldsymbol{\lambda}\}} \mathrm{Tr}_{\mathcal{H}_{out}} \left(\hat{P}^{\boldsymbol{\lambda}}_{\mathcal{H}_{out}} |\psi\rangle\langle\psi|\right) \equiv \sum_{\{\boldsymbol{\lambda}\}} \rho^{\boldsymbol{\lambda}}_{in}. \tag{100}$$

Note that the operators $\rho^{\boldsymbol{\lambda}}_{in}$ do not have a unit trace. Define a density matrix $\tilde{\rho}^{\boldsymbol{\lambda}}_{in}$ with a unit trace, and a probability of sector $p_{\boldsymbol{\lambda}}$:

$$\rho_{in} = \sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} \tilde{\rho}^{\boldsymbol{\lambda}}_{in}. \tag{101}$$

Also, define the trace over a sector as:

$$\mathrm{Tr}_{\mathcal{H}^{\boldsymbol{\lambda}}} = \mathrm{Tr}_{\mathcal{H}}\left(\hat{P}^{\boldsymbol{\lambda}}_{\mathcal{H}} \left|\psi\right\rangle \left\langle\psi\right|\right) \tag{102}$$

for both $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$. Note that for any gauge-invariant state $|\psi_1\rangle$, and non-gauge invariant state $|\psi_2\rangle$, their inner product $\langle\psi_1|\psi_2\rangle$ is 0, so in Eq. (100) only the gauge-invariant states contribute to $\mathrm{Tr}_{\mathcal{H}_{out}} |\psi\rangle \langle\psi|$. Additionally, operators of form $|\phi_1\rangle \langle\phi_2|$ where $|\phi_1\rangle, |\phi_2\rangle$ are gauge-invariant states with different values of electric flux at at least 1 vertex, break gauge invariance. Hence $\rho^{\boldsymbol{\lambda}}_{in}$ only contains non-zero matrix elements only for states in $\mathcal{H}^{\boldsymbol{\lambda}}_{in}$. The entanglement entropy can then be expressed in terms of sector probabilities:

$$\begin{aligned}
S &= -\mathrm{Tr}_{\mathcal{H}_{in}}(\rho_{in} \log \rho_{in}) \\
&= -\sum_{\{\boldsymbol{\lambda}\}} \mathrm{Tr}_{\mathcal{H}_{ginv,in}} \left(\rho^{\boldsymbol{\lambda}}_{in} \log \rho^{\boldsymbol{\lambda}}_{in}\right) \\
&= -\sum_{\{\boldsymbol{\lambda}\}} \mathrm{Tr}_{\mathcal{H}^{\boldsymbol{\lambda}}_{ginv,in}} \left(p_{\boldsymbol{\lambda}} \tilde{\rho}^{\boldsymbol{\lambda}}_{in} \log \left(p_{\boldsymbol{\lambda}} \tilde{\rho}^{\boldsymbol{\lambda}}_{in}\right)\right) \\
&= -\sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} \log p_{\boldsymbol{\lambda}} - \sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} \mathrm{Tr}^{\boldsymbol{\lambda}}_{\mathcal{H}_{ginv,in}} \tilde{\rho}^{\boldsymbol{\lambda}}_{in} \log \tilde{\rho}^{\boldsymbol{\lambda}}_{in} \\
&= H + \sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} S_{\rho^{\boldsymbol{\lambda}}}
\end{aligned} \tag{103}$$

This is valid for any state, so in particular it will also be valid for the ground state of the Kogut-Susskind Hamiltonian, Eq. (52). Note that this is exactly the same expression as obtained using the algebra of observables in Eq. (50), so at least for Abelian gauge theories both methods are consistent. The above calculation can be easily generalised to other Abelian gauge theories, e.g. the $U(1)$ and $\mathbb{Z}_n$ calculations have been pursued in [13].

6 Edge states approach

6.1 Key idea: minimal embedding of $\mathcal{H}_{ginv}$

This section follows the approach of Donnelly [10, 11].

Problems with decomposing the gauge-invariant Hilbert space into a tensor product in lattice gauge theories arise because gauge-invariance imposes Gauss-law type constraints on the boundary. As in Section 5, one of the ways to circumvent it is to embed $\mathcal{H}_{ginv}$ into a larger Hilbert space, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. A question arises: is there a *minimal* way to do so? For example, is there a minimum-dimensional Hilbert space in which $\mathcal{H}_{ginv}$ can be embedded, and which can be decomposed into a tensor product?

The answer to this question is the edge states construction. This method extends the Hilbert space only on the boundary, cutting the links which cross the boundary into two separate links. Then, an embedding of states in $\mathcal{H}_{ginv}$ into $\mathcal{H}$ is constructed, which allows for calculation of the entanglement entropy.

The construction is the same for every discrete or compact gauge group, both Abelian and non-Abelian. It uses the concept of a spin network, which gives a universal basis for any lattice gauge theory.

6.2 Definitions

6.2.1 Spin network basis for $\mathcal{H}_{ginv}$

The **spin network basis** is an orthonormal basis for $\mathcal{H}_{ginv}$. Consider a lattice with directed edges and a gauge group G. A spin network state, i.e. element of the spin network basis, consists of:

- an irreducible representation $r \in R_{all}$ assigned to each link, where R_{all} is the set of all irreducible representations of G.
- an intertwiner $i_n \in I_{all}$ assigned to each node, where I_{all} is the set of all possible intertwiners.

For brevity we denote:

- R as the set of all current assignments of representations to links,
- I as the set of all current assignments of intertwiners to vertices.

An **intertwiner** is a gauge-invariant map between incoming and outgoing representation spaces at each vertex:

$$i_n : \left(\bigotimes_{l:t(l)=n} r_l \right) \to \left(\bigotimes_{l:s(l)=n} r_l \right). \tag{104}$$

The intertwiners are chosen orthonormal in the inner product:

$$\langle i_1, i_2 \rangle = \mathrm{tr}(i_1 i_2^\dagger). \tag{105}$$

Together, these elements uniquely define a spin network state:

$$|S\rangle = |R, I\rangle . \tag{106}$$

Evaluating $|S\rangle$ on an assignment $[U_l] \in G$ to links yields:

$$\langle S|U\rangle = \left(\bigotimes_{l\in L} \sqrt{\dim(r_l)} r_l(u_l) \right) \circ \left(\bigoplus_{n\in N} i_n \right). \tag{107}$$

Any linear functional $|\psi\rangle \in \mathcal{H}_{ginv}$ can be expressed in the spin network basis:

$$|\psi\rangle = \sum_S \psi(S) \, |S\rangle = \sum_{R,I} \psi(R, I) \, |R, I\rangle . \tag{108}$$

Dividing the lattice into two spatial regions A and B, i.e. sets of vertices, we can decompose the notation into:

$$|S\rangle = |R_A, R_B, R_\partial, I_A, I_B\rangle , \tag{109}$$

where R_A, R_B are the representations assigned to links with both ends in A (L_A) and B (L_B) respectively, and R_∂ are the representations assigned to links on the boundary (L_∂), i.e. with one end in A and one end in B.

6.2.2 Spin network basis for $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

The minimally extended Hilbert space $\mathcal{H}$ is constructed by dividing each boundary link $l \in L_\partial$ into $l_A \in L_{\partial A}$ and $l_B \in L_{\partial B}$, and inserting a new node $n \in \partial N$ exactly on the boundary. The new set of links is

$$L' = L_A \cup L_B \cup L_{\partial A} \cup L_{\partial B}. \tag{110}$$

The new Hilbert space $\mathcal{H}_A$ is defined as the set of all linear functionals on the assignments $[U_l]$, where $l \in L_A \cup L_{\partial A}$. The assignment of group elements to links in L', $[U_l]$, can be mapped to an assignment of links in L:

$$\pi : U_{\partial A} \times U_{\partial B} \to U_\partial \tag{111}$$

by mapping each pair of U_{l_A}, U_{l_B} assigned to the boundary links connected to the same boundary node to their product $U_{l_B} U_{l_A}$, as in Figure 8.

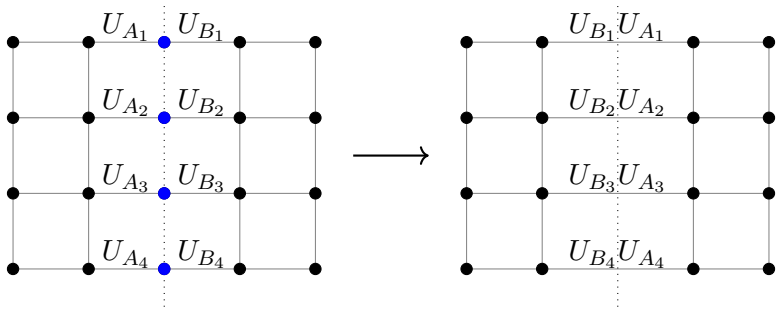

Figure 8: Mapping π of the assignment $[U_{l'}]$ on the extended lattice to the assignment $[U_l]$ on the original lattice, as in Eq. (111).

Any linear functional $|\psi\rangle \in \mathcal{H}_{ginv}$ can then be embedded in $\mathcal{H}_A \otimes \mathcal{H}_B$ using the pullback of π:

$$\pi^* |\psi\rangle ([U_{l_A}], [U_{l_B}]) = |\psi\rangle ([U_l]), \tag{112}$$

with $[U_l]$ constructed using products of boundary links as in Figure 8.

What are the spin network bases for $\mathcal{H}_A$ and $\mathcal{H}_B$? For gauge-invariant states, the representation assigned to connected boundary links in $L_{\partial A}$ and $L_{\partial B}$ are the same. Naively, one may expect the basis for $\mathcal{H}_A$ to be fully specified by $|R_A, R_\partial, I_A\rangle$. However, the vectors in R_∂ cannot be contracted with intertwiners in B anymore, so when evaluating $|S\rangle$ on a assignment $[U_{l_A}]$ there will be leftover uncontracted indices. Hence, a set of vectors $M = m_l : l \in L_\partial$ is needed on the boundary to contract with them and yield a complex scalar $\langle S|U\rangle$. Therefore, a basis state for $\mathcal{H}_A$, also called an **open spin network state**, is specified by:

$$|S_A\rangle = |R_A, R_\partial, I_A, M\rangle . \tag{113}$$

One can map any gauge-invariant spin network state $|S\rangle$ to a state in $\mathcal{H}_A \otimes \mathcal{H}_B$ using the embedding Eq. (112). First, insert a node $n_l \in \partial N$ on each boundary link. Then, assign the identity $\mathbb{1} \in GL(W_{r_l})$ as the intertwiner i_n, where W_{r_l} is the vector space of the representation assigned to the boundary link l. This is possible, as the nodes n_l all have exactly 1 incoming and 1 outgoing link. After splitting, the incoming link is in representation r_l, and the outgoing link - in $\bar{r}_l$. The identity

can be thus decomposed using an orthonormal basis for W_{r_l} and its dual:

$$i_n = \frac{1}{\sqrt{\dim(r_l)}} \sum_{j=1}^{\dim(r_l)} |e_j\rangle \langle e_j| . \tag{114}$$

At any pair of boundary links, incoming l_A and outgoing l_B, there is a representation of G.

$$r_{l_A} : G \to (W_{r_l} \to W_{r_l}) \tag{115}$$
$$r_{l_B} : G \to (\bar{W}_{r_l} \to \bar{W}_{r_l}) \tag{116}$$

Evaluated on a particular assignment $[U_l]$, one obtains the following operators:

$$r_{l_A}(U_l) : \sum_{a=1}^{\dim(r_l)} \sum_{b=1}^{\dim(r_l)} m_{ab} |e_a\rangle \langle e_b| \tag{117}$$
$$r_{l_B}(U_l) : \sum_{a=1}^{\dim(r_l)} \sum_{b=1}^{\dim(r_l)} m^*_{ab} |e_b\rangle \langle e_a| . \tag{118}$$

Composing Eq. (117) and Eq. (118) with the intertwiner from Eq. (114) yields:

$$r_{l_A}(U_{l_A}) \circ i_n \circ r_{l_B}(U_{l_B}) = \frac{1}{\sqrt{\dim(r_l)}} \sum_{a=1}^{\dim(r_l)} \sum_{j=1}^{\dim(r_l)} m_{aj} m^*_{aj} |e_a\rangle \langle e_a| . \tag{119}$$

This is a superposition of tensor product states, where all information about region A is within the vectors $m_{aj} |e_a\rangle$, and for B - within $m^*_{aj} \langle e_a|$. Using the open spin network notation, the embedding is:

$$\pi^*(|S\rangle) = \left(\prod_{p=1}^{P} \frac{1}{\sqrt{\dim(r_l)}} \right) \sum_{m_j} |S_A, m_j\rangle \otimes |S_B, m_j\rangle . \tag{120}$$

Inserting an identity intertwiner at each boundary link yields:

$$\pi^*(|S\rangle) = \prod_{l \in L_{\partial A}} \frac{1}{\sqrt{\dim(r_l)}} \sum_{\{m_l\}} |S_A\rangle \otimes |S_B\rangle , \tag{121}$$

where

$$|S_A\rangle = |R_A, R_\partial, I_A, M\rangle , \qquad |S_B\rangle = |R_B, R_\partial, I_A, M^*\rangle , \tag{122}$$

and M is the set of all tuples $(m_1, ...m_N)$ on all boundary nodes with $m_l \in \{1, ..., \dim r_l\}$.

6.3 Reduced density matrix in $\mathcal{H}_A$

The reduced density matrix is obtained using Eq. (108) and Eq. (121):

$$\rho_A = \sum_{\substack{R_A, R'_A, I_A, I'_A, \\ R_B, R'_B, I_B, I'_B, \\ R_\partial, M}} \frac{\psi(S)\psi(S')^*}{\prod_{l \in L_{\partial A}} \dim(r_l)} |S_A\rangle \langle S'_A| . \tag{123}$$

The aim is to rewrite ρ_A as a direct sum of density matrices in Hilbert spaces corresponding to different sets of boundary representations:

$$\rho_A = \bigoplus_{R_\partial} p(R_\partial)\rho'_A(R_\partial). \tag{124}$$

This will enable us to decompose the entanglement entropy in a way similar to Section 4. Note that gauge-invariance implies only the entries with the same M, M' and $\partial R, \partial R'$ are non-zero in ρ_A. Decomposing further yields:

$$|S_A\rangle = |R_\partial\rangle \otimes |M\rangle \otimes |R_A, I_A\rangle \tag{125}$$
$$|S'_A\rangle = |R_\partial\rangle \otimes |M\rangle \otimes |R'_A, I'_A\rangle . \tag{126}$$

Substituting into Eq. (123)

$$\rho_A = \sum_{R_\partial} |R_\partial\rangle \langle R_\partial| \otimes \left(\sum_M \frac{|M\rangle \langle M|}{\prod_{l\in L_{\partial A}} \dim(r_l)} \right) \otimes \sum_{\substack{R_A, R'_A, R_B \\ I_A, I'_A, I_B}} \psi(S)\psi(S') |R_A, I_A\rangle \langle R'_A, I'_A| . \tag{127}$$

The third term can be normalised to obtain a unit-trace density matrix on the Hilbert space spanned by $|R_A, I_A\rangle$:

$$\rho_A = \sum_{R_\partial} p(R_\partial) |R_\partial\rangle \langle R_\partial| \otimes \left(\sum_M \frac{|M\rangle \langle M|}{\prod_{l\in L_{\partial A}} \dim(r_l)} \right) \otimes \tilde{\rho}_A(R_\partial) \tag{128}$$

with

$$\tilde{\rho}_A(R_\partial) = \sum_{\substack{R_A, R'_A, R_B \\ I_A, I'_A, I_B}} \frac{\psi(S)\psi(S')}{p(R_\partial)} |R_A, I_A\rangle \langle R'_A, I'_A| . \tag{129}$$

As the same state appears in the bra and ket in $|R_\partial\rangle \langle R_\partial|$, this is a direct sum Hilbert space:

$$\rho_A = \bigoplus_{R_\partial} p(R_\partial) \left[\left(\sum_M \frac{|M\rangle \langle M|}{\prod_{l\in L_{\partial A}} \dim(r_l)} \right) \otimes \sum_{\substack{R_A, R'_A, R_B \\ I_A, I'_A, I_B}} \frac{\psi(S)\psi(S')}{p(R_\partial)} |R_A, I_A\rangle \langle R'_A, I'_A| \right] . \tag{130}$$

The second term is dependent only on diagonal terms $|M\rangle \langle M|$, and thus proportional to the identity of all boundary link representations.

$$\rho_A = \bigoplus_{R_\partial} p(R_\partial) \left[\left(\frac{\mathbb{1}_{r_l}}{\dim(r_l)} \right) \otimes \tilde{\rho}_A(R_\partial) \right] . \tag{131}$$

6.4 Entanglement entropy for states in the spin network basis

Entanglement entropy of a weighted direct sum of density matrices is, as in Eq. (50):

$$S\left(\bigoplus_n p_n \rho_n \right) = -\sum_n p_n \log p_n - \sum_n p_n \rho_n \log \rho_n = H(p_n) + \sum_n p_n S(\rho_n). \tag{132}$$

Entropy of a tensor product state is additive:

$$S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2). \tag{133}$$

The entanglement entropy of a maximally mixed state of dimension d, i.e. proportional to the identity, is:

$$S\left(\frac{\mathbb{1}_d}{d}\right) = \log d. \tag{134}$$

Using these properties, the entropy of Eq. (131) becomes:

$$\begin{aligned} S(\rho_A) &= -\sum_{R_\partial} p(R_\partial) \log p(R_\partial) - \sum_{R_\partial} p(R_\partial) \sum_{l \in L_{\partial A}} \log \dim(r_l) - \sum_{R_\partial} p(R_\partial)\ S(\tilde{\rho}_A(R_\partial)) \\ &= H(p(R_\partial)) + \sum_{l \in L_{\partial A}} \langle \log \dim(r_l) \rangle + \langle S(\tilde{\rho}_A(R_\partial)) \rangle. \end{aligned} \tag{135}$$

6.5 $\mathbb{Z}_2$ theory

6.5.1 Ground state in the spin network basis

In the spin network basis, the ground state can be written as:

$$|\Psi\rangle = \frac{1}{\mathcal{N}} \sum_S |S\rangle \tag{136}$$

The possible representations assigned to edges in any spin network state are the only 2 representations of $\mathbb{Z}_2$:

1. Trivial representation, labelled 1. $R_1 := 1 \to 1,\ -1 \to 1$.

2. Alternating representation, labelled -1. $R_{-1} := 1 \to 1,\ -1 \to -1$.

Within a fixed superselection sector, i.e. for a fixed set of boundary representations R_∂, the state $\rho_A(R_\partial)$ is just:

$$|\rho_A\rangle \propto \sum_{S_A, S'_A} |S_A\rangle \langle S'_A| = \sum_{S_A} |R_A, I_A\rangle \langle R'_A, I'_A| = |\psi\rangle \langle \psi|, \tag{137}$$

with $|\psi\rangle = \sum_{R_A, I_A} |R_A, I_A\rangle$.

6.5.2 Calculating entropy

Eq. (135) gives 3 terms for entanglement entropy. The first term vanishes because the matrix in Eq. (137) is pure. The second term vanishes because all possible representations of $\mathbb{Z}_2$ are 1-dimensional.

The third term naively arises from equally 2 possible representations for each link, so that the entanglement entropy is $n \log 2$, where n is the number of nodes on the boundary. However, gauge invariance implies there must be an even number of links with assigned representation -1.

To see that, suppose that on the lattice there is a connected path of links with assigned -1 representations which is not a loop. For simplicity, consider a grid made up of 4 links aligned in a

square only. Then, consider an assignment of variables $U_1, U_2, U_3, U_4 \in G$, and another assignment $U_l' \in G$ obtained by a gauge transformation. As an example, take all U_l to be 1, and take U_l' to be the transformation by $g_{V_1} = -1$ at the vertex V_1.

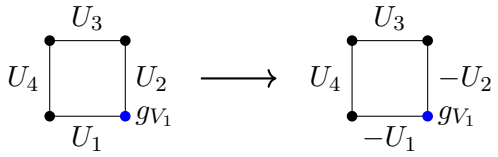

Figure 9: Gauge transformation by $g_{V_1} = -1$ at the vertex V_1 on a simple square lattice with a configuration of fields U_l.

Gauge invariance implies that under the representations R_l the variables U_l must be mapped to an assignment U_l' beloning to the same equivalence class under all possible gauge transformations. Consider a basis spin state with the following representations R:

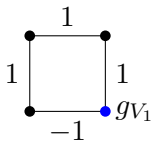

Figure 10: Example basis state with assigned representations to the links. The assignment is denoted R. Note that these are not the variables $[U_l]$.

As all intertwiners were defined to be gauge-invariant and the maps are all linear, the spin network functional is gauge invariant iff the mapping of configurations $\{U_l\}$ under the representation R_l is gauge-invariant. Hence, we will only consider what happens when we act with R on $\{U_l\}$.

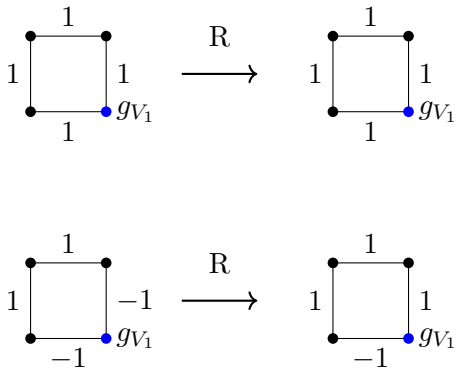

Figure 11: Mapping of the 2 states U_l, U_l' belonging to the same equivalence class under gauge transformations by R

Note that in Figure 11, the 2 resultant configurations do not belong to the same equivalence class. Repeating the same process with R such that $R_1 = R_2 = R_3 = R_4 = 1$ or $R_1 = R_2 = R_3 = R_4 = -1$

shows that the configurations obtained do belong to the same equivalence class. In general, for 'open string' asignments R, i.e. ones where there exists an open path with representation -1 on all links in the path, one can always insert a gauge transformation on the vertex connected to an odd number of links with representation -1, and prove that the mapping under R is not gauge-invariant.

Thus, gauge invariance implies that in each basis state -1 representations can be assigned only to closed loops. As a result, there must be an even number of links with a -1 representation on the boundary. Hence, the number of choices is 2^{n-1} for every connected component of the boundary, as the value of the last link is decided by the parity of -1 links. In total, for k connected components of the boundary the number of choices is 2^{N-k}, where $N = \sum_k n_k$. Finally, the entanglement entropy is:

$$S_A = (N-k)\log 2, \tag{138}$$

where N is the length of the boundary. This is consistent with the CSH and GST methods. An extension of this method to $U(1)$ can be found in [11], which is consistent with a calculation for $U(1)$ using the replica trick in [28].

7 Non-Abelian theories

7.1 Overview

What happens in non-Abelian theories? There are several avenues here. Firstly, the decomposition Eq. (135) is already valid for non-Abelian groups as well, as the approach pursued in Section 6 did not assume Abelian gauge groups. Additionally, the extended Hilbert space construction from Section 5 can be generalised to non-Abelian groups, with the only difference being some mathematical subtleties regarding left and right multiplication and inverse link operators - we will see this in Section 7.2.

A more tricky, and perhaps more physically relevant question is: what happens for specific physical states, or in the presence of matter? The approaches of Section 6 and Section 5 do not actually *choose* a state, so for any meaningful comments about e.g. the Standard Model or topological matter we need to select a particular state in the Hilbert space of the theory. As famously known from QCD, even for $SU(N)$ the spectrum of states can be complicated. Thus, the state of the art is calculating entanglement entropies at the strong and the weak coupling limits in the Kogut-Susskind Hamiltonian.

7.2 SU(2) entanglement entropy using the GST method

In $SU(2)$ lattice gauge theory, the variables on the links (ij), U_{ij}, are elements of the $SU(2)$ group, so in a chosen basis, they can be written as $(U_{ij})^a{}_b$. We will suppress the matrix indices. As before, in the quantum theory we can promote U_{ij} to operators $\hat{U}_{ij} \in \mathcal{H}_{ij}$, and choose a basis of eigenstates of the $\hat{U}_{ij}$:

$$\hat{U}_{ij}\,|U_{ij}\rangle = U_{ij}\,|U_{ij}\rangle\,. \tag{139}$$

Gauge transformation variables are attached to vertices. Every set of transformations can be expressed as a series of infinitesimal transformations, so without loss of generality we can take the transformation variables $e^{\frac{i}{2}\omega^A J^A_{ij}} \in SU(2)$ at a vertex n which fulfil:

$$e^{\frac{i}{2}\omega^A J^A_{ij}}\,|U_{ij}\rangle = |\left(\mathbb{1} + \frac{i}{2}\omega^A\sigma^A\right)U_{ij}\rangle\,, \tag{140}$$

where σ^A are the Pauli matrices, and ω^A, $A = 1, 2, 3$ are infinitesimal parameters. The main difference from Abelian theories is that J^A_{ij} are the generators of rotations for the *directed* link (ij) going $from$ vertex i, and the corresponding generator for the link (ji) going *into* vertex i will obey:

$$e^{\frac{i}{2}\omega^A J^A_{ji}} |U_{ij}\rangle = |U_{ij} \left(\mathbb{1} - \frac{i}{2}\omega^A \sigma^A \right) \rangle \,. \tag{141}$$

Notice that in the second case the orientation of J^A_{ji} in the gauge transformation is opposite to that of $|U_{ij}\rangle$. Importantly, their components all commute:

$$[\hat{J}^A_{ij}, \hat{J}^B_{ji}] = 0, \tag{142}$$

as one of them acts from the left, and the other from the right. The rest of the commutation relations follow from the $SU(2)$ algebra:

$$[J^A_{ij}, J^B_{ij}] = i\varepsilon^{ABC} J^C_{ij}; \quad [J^A_{ji}, J^B_{ji}] = i\varepsilon^{ABC} J^C_{ji}. \tag{143}$$

Operators acting on 2 different links commute trivially as they act on different Hilbert subspaces in the tensor product of all $\mathcal{H}_{ij}$.

Following the discussion in Section 5, we can divide the physical gauge-invariant states in the superselection sectors, using eigenstates of operators at the boundary. As in Section 7.2, the generators are e.g. $(J^A_{12} + J^A_{13})^2$, $\quad (J^A_{45})^2$.

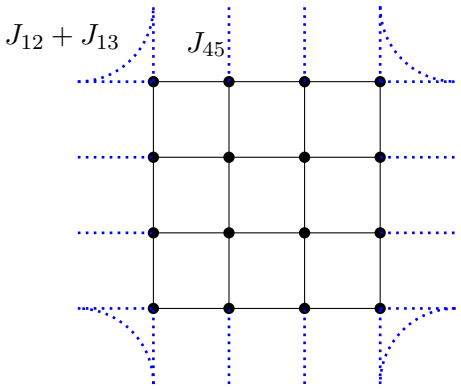

Figure 12: Generators of the electric centres, labelled by the connected nodes and highlighted in blue. The sums of two generators attached to each corner are a single generator.

From Gauss's law at each boundary vertex, we have:

$$(J_{\text{out},v})^2 = (J_{\text{in},v})^2, \tag{144}$$

$$J^A_{\text{out},v}|\psi\rangle = -J^A_{\text{in},v}|\psi\rangle, \quad A = 1, 2, 3. \tag{145}$$

Note that it is not necessary to additionally specify a single component, J^3_{ij}. This is because from Eq. (145), $J^A_{\text{out},v}$ can be rewritten in terms of $J^A_{\text{in},v}$, so the reduced density matrices in each superselection sector obey:

$$[J^A_{\text{in},v},\ \rho^{\boldsymbol{\lambda}}_{in}] = [-J^A_{\text{out},v},\ \rho^{\boldsymbol{\lambda}}_{in}] \ = \ 0, \tag{146}$$

The eigenvalues will have no non-trivial dependence on J^A on the boundary, and same follows for the entanglement entropy. Choosing the sectors based only on values of J_{ij}^2 is therefore sufficient to calculate the entanglement entropy using the GST method. A note for later: we will deliberately **not** name the sectors here *superselection sectors*.

Using exactly the same reasoning as before, the eigenvalues of total outgoing angular momentum at the boundary entirely specifies the sector within $\mathcal{H}_{out}$. Similarly, one can use the opposite links, e.g. $(J_{21} + J_{31})^2, J_{54}^2$ to specify the values of angular momentum going into the region, thus specifying a region within $\mathcal{H}_{in}$. The generators J_{45}^2 and J_{54}^2 are no longer equal, as they were in $\mathbb{Z}_2$ theory where both were simply σ_x, but they still commute, which is all we need to express both Hilbert spaces as direct sums over all possible $\boldsymbol{\lambda}$, which are the simultaneous eigenvalues of J_{ij}^2, J_{ji}^2 at each boundary vertex (or sums thereof on the corners). The rest of the discussion goes exactly as in the $\mathbb{Z}_2$ case. Each sector is denoted by its $\boldsymbol{\lambda}$, and the final entanglement entropy is given by

$$S = -\sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} \log p_{\boldsymbol{\lambda}} - \sum_{\{\boldsymbol{\lambda}\}} p_{\boldsymbol{\lambda}} \mathrm{Tr}^{\boldsymbol{\lambda}}_{\mathcal{H}_{ginv,in}} \tilde{\rho}^{\boldsymbol{\lambda}}_{in} \log \tilde{\rho}^{\boldsymbol{\lambda}}_{in} \tag{147}$$

analogously to Eq. (103).

Here is a problem - are those sectors actually superselection sectors, i.e. can we get from any state to any other within that sector using only gauge-invariant operators? As follows from Gauss' law in Eq. (145) any gauge-invariant operator local to $\mathcal{H}_{in}$ commutes with $J^3_{\mathrm{in},v}$ with $v \in \partial A$, so they cannot change the value of J^3 for any links. One sector here turns out to be a direct sum of many superselection sectors, specified both by $(J)^2$ and J^3. This is not the case for Abelian groups.

As a result, not all entanglement in the quantum part of Eq. (147) can be *distilled*, i.e. turned into Bell pairs by local, gauge-invariant operations and classical communication [27]. This means that it cannot be used as a resource in processing quantum information. Differences between this operational view of entanglement, often used in quantum information contexts, and the entanglement entropy in gauge theories will be further discussed in Section 8.

7.3 Strong coupling regime

In the strong coupling limit of $SU(N)$, it is possible to perturbatively calculate the entropy of n^{th} order excitations, which, in order to preserve gauge invariance, have a form of closed loops. In the ground state for $g \to \infty$, there are no excitations and the overall state is a tensor product of individual link states. In the spin network basis Section 6, the ground state is:

$$|\Omega\rangle = \bigotimes_l |\mathbf{1}\rangle\,, \tag{148}$$

where $\mathbf{1}$ denotes the trivial representation. In the 'position' basis of group variables, this is the maximal superposition on each link:

$$|\Omega\rangle = \bigotimes_l \int_G dU\, |U\rangle_l\,, \tag{149}$$

where dU is the normalised Haar measure on the group manifold. For our purposes, it is sufficient to think of this as integration over all group elements. The entropy of such state is 0 for any

region, as it is a pure tensor product state. Let us try to calculate the contribution of first-order excitations. An excitation on a single link is:

$$|r\rangle_l = \int_G U^r \, |U\rangle \, . \tag{150}$$

These states span the entire extended Hilbert space [19, 25], so we can use it for partial tracing. The gauge-invariant excitations diagonal in this basis will be the Wilson loops. For first-order contributions, the lowest-energy excitations will matter, so we will restrict ourselves to the fundamental and antifundamental representations:

$$|\mathrm{f}_{\alpha\beta}\rangle_l = \sqrt{N} \int_G dU \; U_{\alpha\beta} \, |U\rangle_l \; , \quad |\bar{\mathrm{f}}_{\alpha\beta}\rangle_l = \sqrt{N} \int_G dU \; (U^{-1})_{\beta\alpha} \, |U\rangle_l . \tag{151}$$

A closed loop of links, $\Gamma = (l_1, ..., l_n)$, assuming all of them are traversed in the direction of their orientation, is:

$$|\Box_\Gamma\rangle = \int_G dU_1 \dots dU_n \, \mathrm{Tr}_n \left(U_1 \dots U_n \, |U_1\rangle_{l_1} \dots |U_n\rangle_{l_n} \right) = N^{-\frac{n}{2}} \; |\mathrm{f}_{\alpha_1\alpha_2}\rangle_{l_1} \otimes ... \otimes |\mathrm{f}_{\alpha_n\alpha_1}\rangle_{l_n} \, , \tag{152}$$

where the trace comes from the fact that Γ is a closed loop. If a link is traversed in the opposite direction, one should replace U with $(U^{-1})^T$ in the expression. The first-order corrected state can be then written as:

$$\begin{aligned} |\Omega_{g>>1}\rangle &= \left(1 - \frac{N_p}{\lambda^2}\right) |\Omega\rangle + \frac{1}{\lambda} \sum_p \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) \\ &= \left(1 - \frac{N_p}{\lambda^2}\right) |\Omega\rangle + \frac{1}{\lambda} \sum_{p \notin \partial A} \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) + \frac{1}{\lambda} \sum_{p \in \partial A} \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) , \end{aligned} \tag{153}$$

where $|\Box\rangle$ denotes a **plaquette**, the smallest possible loop on a square grid, i.e. a 4-link square, and N_P is the finite number of plaquettes on the lattice. For perturbative methods to hold, we will assume $\lambda >> N_p$ [28]. The sum was split into plaquettes touching the boundary, and those entirely inside or outside. The latter states will all be in the same sector, $\boldsymbol{\lambda}_0 = (0, ..., 0)$, as no excited links lie on the boundary. The boundary plaquette states are all in different sectors from each other, and none of them are in $\boldsymbol{\lambda}_0$. Hence, there will be no off-diagonal terms mixing the boundary and non-boundary plaquettes. The density matrix of this state is:

$$\begin{aligned} \rho_{g>>1} = &\left[\left(1 - \frac{N_p}{\lambda^2}\right) |\Omega\rangle + \frac{1}{\lambda} \sum_{p \notin \partial A} \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) \right] \left[\left(1 - \frac{N_p}{\lambda^2}\right) \langle\Omega| + \frac{1}{\lambda} \sum_{p \notin \partial A} \left(\langle\Box_p| + \langle\bar{\Box}_p| \right) \right] \\ &+ \left[\frac{1}{\lambda} \sum_{p \in \partial A} \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) \right] \left[\frac{1}{\lambda} \sum_{p \in \partial A} \left(\langle\Box_p| + \langle\bar{\Box}_p| \right) \right] \\ \equiv &\, |\Phi\rangle \langle\Phi| + \left[\frac{1}{\lambda} \sum_{p \in \partial A} \left(|\Box_p\rangle + |\bar{\Box}_p\rangle \right) \right] \left[\frac{1}{\lambda} \sum_{p \in \partial A} \left(\langle\Box_p| + \langle\bar{\Box}_p| \right) \right] \end{aligned} \tag{154}$$

Within the $\boldsymbol{\lambda}_0$ sector, the state is pure, so there will be no quantum entropy contribution from off-boundary states. From Eq. (153) we can read off the probability of a single plaquette excitation, i.e. $\frac{1}{\lambda}$. The probability of a state being in the $\boldsymbol{\lambda}_0$ sector is thus $1 - \frac{2|\partial A|}{\lambda}$, considering fundamental and antifundamental excitations for each plaquette.

Here we will use the formula for entanglement entropy from Section 6. Importantly, the region A is defined there as set of *vertices*, so its boundary cuts through links. The classical contribution is:

$$\begin{aligned} H &= -\sum_{\boldsymbol{\lambda}} p_{\boldsymbol{\lambda}} \log p_{\boldsymbol{\lambda}} \\ &= -\left(1-\frac{2|\partial A|}{\lambda^2}\right)\log\left(1-\frac{2|\partial A|}{\lambda^2}\right) - \sum_{i=1}^{2|\partial A|}\frac{1}{\lambda^2}\log\frac{1}{\lambda^2} \\ &= -\left(1-\frac{2|\partial A|}{\lambda^2}\right)\log\left(1-\frac{2|\partial A|}{\lambda^2}\right) - 2|\partial A|\frac{1}{\lambda^2}\log\frac{1}{\lambda^2} \\ &\approx -\left(1-\frac{2|\partial A|}{\lambda^2}\right)\left(-\frac{2|\partial A|}{\lambda^2}\right) - 2|\partial A|\frac{1}{\lambda^2}\log\frac{1}{\lambda^2} \\ &= \frac{2|\partial A|}{\lambda^2}\left(1-\log\frac{1}{\lambda^2}\right). \end{aligned} \tag{155}$$

The contribution from representation dimensions is:

$$\sum_{i=1}^{2|\partial A|\times 2}\frac{1}{\lambda^2}\log(N) = \frac{2|\partial A|}{\lambda^2}\log(N^2), \tag{156}$$

where N is from the $SU(N)$ symmetry, and the additional $\times 2$ comes from the fact that each plaquette cuts through the boundary twice. The total entanglement entropy is:

$$S = \frac{2|\partial A|}{\lambda^2}\left(1-\log\frac{1}{\lambda^2}+\log(N^2)\right). \tag{157}$$

In the strong coupling limit, $\lambda \to \infty$, the prefactor $\frac{1}{\lambda^2}$ dominates the logarithmic growth, so the entropy vanishes as expected. When λ is finite, but still in the strong coupling regime, we can see that an area law emerges from the perturbative calculation.

7.4 Weak coupling regime

7.4.1 Weak coupling state looks like decoupled Goldstone bosons

In the weak coupling regime, the situation is a bit more complicated. Naively, one may expect that the ground state of zero coupling may look like all possible Wilson loop excitations. However, as here the gauge group $SU(N)$ is continuous, and has infinitely many representations, there are infinitely many such single-excitation states, and so the entanglement entropy will naively appear to be infinite.

To see this in more detail, let us first choose a convenient gauge. In 2D lattice field theories, one can fix the gauge by choosing a suitable subset of links l, called the 'dead' links, and set $U_l = \mathbb{1}$ for all of them. Then, the $U_{l'}$ on rest of the links can take any value. A useful choice is the **axial gauge**, where the dead links are: all x-oriented links and every other y-oriented column. An analogous construction can be done in $d > 2$ by iteration on dimensions. The result is that only 1 link per plaquette is living. The resulting Hilbert space on the living links, $\mathcal{H}$ is spanned by $|U_l^\star\rangle$, and is isomorphic to the original $\mathcal{H}_{ginv}$. Hence, all gauge-invariant operators on $\mathcal{H}_{ginv}$ have can be mapped to operators on $\mathcal{H}^\star$.

The hypothetical state will then be:

$$|\psi\rangle \propto \prod_{l^\star} \left(\sum_{r_l} d_{r_l} |r_l\rangle \right), \tag{158}$$

where the product runs over all living links. However, even after gauge fixing, the living links will have infinite possible excitations corresponding to different representations. Hence, the entanglement entropy appears to be infinite.

But this naive prescription is wrong. To solve this properly, we would have to regularise the group manifold with a short-distance cutoff. The detail of this can be found in [28], but the upshot is that the resulting Hamiltonian looks like $\dim(G)$ decoupled harmonic oscillators, or non-compact photons, sometimes called the Coulomb phase. We will denote its ground state as $|\text{Coulomb}\rangle$.

The entanglement entropy will then decompose into:

$$S_{\text{Coulomb}} = \dim(G)\, S_{\text{photon}}. \tag{159}$$

7.4.2 Warm-up: toy rotor model

Before the entanglement entropy of the weak coupling Kogut-Susskind Hamiltonian, it is useful to understand another model - the quantum rotor. We will work in the continuum, in a finite volume $V = L^d$. We will later find out that the weak gauge theory limit is mathematically similar, and we will be able to reuse some parts of the rotor solution. The rotor Lagrangian is:

$$L = \frac{\rho_s V_A (\partial_t \hat{n}_A)^2}{2} + \frac{\rho_s V_B (\partial_t \hat{n}_B)^2}{2} - J \hat{n}_A \cdot \hat{n}_B, \tag{160}$$

where ρ_s is just a constant, sometimes called the stiffness, $\vec{n}_A, \vec{n}_b$ are unit vectors describing the orientation of the rotors, and V_A, V_B are the volumes of the subsystems, and J is the coupling constant between the two rotors. Define the average and relative coordinates, $\vec{n}$ and δn by:

$$\begin{aligned} \vec{n}_A &= \vec{n}\sqrt{1 - a^2(\delta_{n_A})^2} + \vec{a} e_\alpha \delta n_\alpha, \quad a = \frac{V_B}{V} \\ \vec{n}_B &= \vec{n}\sqrt{1 - b^2(\delta_{n_A})^2} + \vec{b} e_\alpha \delta n_\alpha, \quad b = -\frac{V_A}{V}. \end{aligned} \tag{161}$$

We expect the fluctuations to be small in general, but shift quicker than the average orientation. Substituting this into the original Lagrangian and assuming: $\delta n \ll n$, $\partial_t \delta n \gg \partial_t \vec{n}$, we get the following approximate Lagrangian:

$$\begin{aligned} L_{approx} &= \frac{\rho_s V}{2} (\partial_t \vec{n})^2 + \frac{\rho_s V_r}{2} (\partial_t \delta n_\alpha)^2 + \frac{J}{2} (\delta n_\alpha)^2 \\ &\equiv L_n + L_{\delta n}, \end{aligned} \tag{162}$$

where $V_r = \frac{V_A V_B}{V_A + V_B}$ is the reduced volume. The average solution is a free spherical rotor, and the fluctuations behave like simple harmonic oscillators. The solution to this is a product of solutions to each part:

$$\psi(\vec{n}, \delta n) = \frac{1}{\sqrt{|S_{N-1}|}} \frac{1}{(\pi \xi^2)^{\frac{N-1}{4}}} \exp\left(-\frac{(\delta n_\alpha)^2}{2\xi^2} \right), \tag{163}$$

with

$$\xi = \sqrt[4]{\frac{1}{\rho_s V_r J}} \sim \sqrt[2]{\frac{1}{\rho_s L^{d-1}}}, \xi \ll 1. \tag{164}$$

Now we can calculate the reduced density matrix, where reduced means we are tracing over the states localised in region B:

$$\rho_A(\vec{n}_A, \vec{n}_{A'}) = \int d\vec{n}_B \; \psi(\vec{n}_A, \vec{n}_B) \; \psi^*(\vec{n}'_A, \vec{n}_B). \tag{165}$$

If $\vec{n}_B - \vec{n}_A$ is large, the contributions will be exponentially suppressed, so to first order we can only take the fluctuations to lie in the tangent plane to $\vec{n}_A$, thus reducing the dimension of the integral to $d-1$. Redefining $\vec{n}_A - \vec{n}_B = \delta n$, $\vec{n}'_A - \vec{n}_B = \delta n + \vec{n}'_A - \vec{n}_A$, we get:

$$\begin{aligned}
\rho_A(\vec{n}_A, \vec{n}'_A) &= \frac{1}{|S_{N-1}|} \frac{1}{(\pi\xi^2)^{\frac{N-1}{2}}} \int d\;\delta n \; \exp\left(-\left(\frac{\delta n}{2\xi^2}\right)^2\right) \; \exp\left(-\left(\frac{(\vec{n}_A - \vec{n}'_A) - \delta n}{2\xi^2}\right)^2\right) \\
&= \frac{1}{|S_{N-1}|} \frac{1}{(\pi\xi^2)^{\frac{N-1}{2}}} \int d\;\delta n \; \exp\left(-\left(\left[\frac{\delta n}{\xi} - \frac{\vec{n}_A - \vec{n}'_A}{2\xi}\right]^2\right)\right) \; \exp\left(\frac{\vec{n}_A - \vec{n}'_A}{4\xi^2}\right) \\
&= \frac{1}{|S_{N-1}|} \exp\left(\frac{\vec{n}_A - \vec{n}'_A}{4\xi^2}\right),
\end{aligned} \tag{166}$$

as the integral is just a Gaussian, so it cancels the prefactor. Instead of directly using this form, we will use a heat kernel method to relate the result above to spin operators. The **heat kernel** of an operator D is defined as:

$$K(t; x, y; D) = \langle x | \exp(-tD) \, | y \rangle \,, \tag{167}$$

which is a solution to the heat conduction equation,

$$(\partial_t + D_x) K(t; x, y; D) = 0. \tag{168}$$

If $D = D_0 = \nabla^2$, the massless heat kernel is well known:

$$K_0 = K(t; x, y; D_0) = (4\pi t)^{\frac{N}{2}} \exp\left(-\frac{(x-y)^2}{4t}\right). \tag{169}$$

More on heat kernel methods can be found in [32], but the equality above will suffice here. On a hypershere, the total angular momentum $\hat{L}^2$ is the Laplacian ∇^2, so we can use Eq. (169) to get:

$$\langle \vec{n} | \exp(-\xi^2 \hat{L}^2) \, | \vec{n}' \rangle = (4\pi\xi^2)^{\frac{N}{2}} \exp\left(-\frac{(x-y)^2}{4\xi^2}\right), \tag{170}$$

where N is the number of dimensions, and

$$\langle \vec{n} | \exp(-k\xi^2 \hat{L}^2) \, | \vec{n}' \rangle = (4\pi k\xi^2)^{\frac{N}{2}} \exp\left(-\frac{(\vec{n} - \vec{n}')^2}{4k\xi^2}\right), \tag{171}$$

where k is an arbitrary non-zero integer. This allows us to relate the density matrix to spin:

$$\rho_A \approx \frac{4\pi\xi^{2\,\frac{N-1}{2}}}{|S_{N-1}|} \exp^{-\xi^2 \hat{L}^2}. \tag{172}$$

Now, we will compute the entanglement entropy using the replica trick - first, we will find the k-th Renyi entropy, and then take its limit as $n \to 1$.

$$\begin{aligned}
\mathrm{Tr}(\rho_A^k) &\approx \frac{(4\pi\xi^2)^{k(N-1)/2}}{|S^{N-1}|^k}\mathrm{Tr}(e^{-k\xi^2\hat{L}_A}) \\
&= \frac{(4\pi\xi^2)^{k(N-1)/2}}{|S^{N-1}|^k}\int d\vec{n}\,\langle\vec{n}|\exp(-k\xi^2\hat{L}^2)\,|\vec{n}\rangle \\
&= \frac{(4\pi\xi^2)^{k(N-1)/2}}{|S^{N-1}|^k}\frac{|S^{N-1}|}{(4\pi\xi^2)k^{(N-1)/2}} \quad \text{using Eq. (171)} \\
&= \frac{(4\pi\xi^2)^{(k-1)(N-1)/2}}{|S^{N-1}|^{k-1}k^{(N-1)/2}}.
\end{aligned} \tag{173}$$

The k-th Renyi entropy is:

$$\begin{aligned}
(S_A)_k &= -\frac{1}{k-1}\log\mathrm{Tr}(\rho_A^k)\frac{N-1}{2} \\
&= \frac{N-1}{2}\log\frac{1}{4\pi\xi^2} + \frac{N-1}{2}\frac{1}{k-1}\log k + \log|S^{N-1}| \\
&\sim \frac{N-1}{2}\log\rho_s L^{d-1} + \text{const as } k \to 1,
\end{aligned} \tag{174}$$

which is logarithmic in the characteristic area associated to the reduced volume of the system in Eq. (164). Note that this is not necessarily equal to the area of the system A - we have not even defined such quantity, but that is alright - the toy rotor calculation has only been done as a mathematical primer for the weak coupling gauge theory. What is important is that a term logarithmic in some area has appeared.

The Lagrangian for the average part of the toy rotor, i.e. L_n from Eq. (162) can be alternatively written in the Hamiltonian formulation:

$$H_n = \frac{L_{total}^2}{2\rho_s V}, \quad L_{total} = \hat{L}_A + \hat{L}_B, \tag{175}$$

where ρ_s is just a constant, sometimes called the stiffness, $\vec{n}_A, \vec{n}_B$ are unit vectors describing the orientation of the rotors, and V is the volume of the system, taken here to be a box. The operator $\hat{L}_{ab}$ is the usual angular momentum operator,

$$L_{ab} = -i\left(n^a\frac{\partial}{\partial n^b} - n^b\frac{\partial}{\partial n^a}\right). \tag{176}$$

Surprisingly, H_n is exactly the operator whose heat kernel turned out to be approximately equal to the density matrix in Eq. (172). This has a deeper underpinning, as the seemingly unrelated operator in *Eq.* (171) is the *entanglement Hamiltonian* is the system, which in this case turns out to be equal to the first-order approximation to the full Hamiltonian. This is the case in the model known as *Anderson tower of states* Hamiltonian. A more detailed description of this and further consistency checks can be found in [20].

The key point is: if a Hamiltonian of a system looks like $c\hat{L}^2$, its entanglement entropy will follow Eq. (174) with appropriately adjusted constants.

7.4.3 The weak coupling ground state and its entanglement entropy

Using our knowledge of the toy rotor model, we can find the first-order entanglement entropy for the weak coupling limit of non-Abelian gauge theories. In [25], it was argued that the effective description of slowly changing gauge fields $\{U_l\}$ on a region V can be captured by a single degree of freedom, $U \in G$, its environment $W \in G$, and possible but weak interactions between them. The effective Lagrangian is:

$$L = \frac{1}{2}c_U \mathrm{Tr}(\partial U^{-1}\partial U) + \frac{1}{2}c_W \mathrm{Tr}(\partial W^{-1}\partial W) - \frac{J}{2}[2N - \mathrm{Tr}(U^{-1}W) - \mathrm{Tr}(W^{-1}U)], \tag{177}$$

where c_U, c_W are the 'kinetic' self-coupling constants, where $c_U \sim |V|$. Notice this is exactly the Yang Mills version of the toy rotor Lagrangian, Eq. (162). The effective coupling between V and the outside, J, will scale like r^{d-2}, where r is the characteristic length of the region V. [20]. Assuming $V \ll V \cup \bar{V}$, the reduced coupling $c_r = c_U c_W/(c_U + c_W)$ will also scale with the volume $|V|$, or with r^d. Analogously to Eq. (164), $\xi^2 \sim \sqrt{c_r J} \sim 1/r^{d-1} \sim g^2/|\partial V|$, where $|\partial V|$ is the size of the boundary, and g^2 is another constant capturing the coupling strength.

The ground state of this Lagrangian is:

$$\psi(U, W) = \frac{1}{\sqrt{\mathrm{Vol}(G)}} \frac{1}{(\pi\xi^2)^{\dim(G)/4}} \exp\left(-\frac{1}{2}\mathrm{Tr}(2N - \mathrm{Tr}(U^{-1}W) - \mathrm{Tr}(W^{-1}U))\right), \tag{178}$$

or in terms of small fluctuations $U = We^{iA^bT^b}$:

$$\psi(A) = \frac{1}{(\pi\xi^2)^{\dim(G)/4}} \exp\left(\frac{(A^b)^2}{2}\right) \tag{179}$$

Using exactly the same reasoning as in the toy model, the first-order contribution to the entanglement entropy in the weak coupling regime is:

$$\Delta S(g) = \frac{\dim(G)}{2} \log\left(\frac{|\partial V|}{g^2}\right). \tag{180}$$

Further resummation of beyond first-order terms was found to yield an area law in [20]. It is possible to redo the weak coupling calculation in full detail without using an effective Lagrangian, keeping all gluon degrees of freedom, writing the solutions using propagators and decomposing into different sectors. This was done in Section 5.3. of [25]. A similar calculation has been done in [24]. The perturbative QFT approach agrees with the effective Lagrangian picture to the first order.

7.5 Adding matter to the SU(N) lattice gauge theory

So far everything we have done applied to *pure gauge* theories, i.e. with no physical matter present. What happens if matter is added to the picture? This is a complicated topic, but a study by Aoki et al. [2] has explained the emergence of various entanglement entropy terms in the presence of scalar matter fields on the vertices. Though the full calculations are beyond the scope of this essay, the framework can be explained in few steps. Aoki et al. focus on a ring-like 1+1-dimensional lattice, consisting of gauge fields $U_{12}, U_{23}, ..., U_{n1}$. Then, matter is added in meson-like pairs of neighbouring vertices, in the form of 1 creation and 1 annihilation operator. For instance, one of the states they consider are:

$$\Psi(U_{ij}, \phi_i) = \frac{1}{\mathcal{N}}[\phi_1^\dagger U_{12} U_{23} U_{34} U_{45} \phi_5^\dagger] \prod_{i-1}^{n_l=7} e^{-\frac{\gamma}{2}\phi_i^\dagger \phi_i}, \tag{181}$$

where $\frac{1}{\mathcal{N}}$ and $\prod e^{-\frac{\gamma}{2}\phi_i^\dagger\phi_i}$ ensure normalisation. In the presence of matter, we now also have to decide if matter is inside or outside, so the 'inside' region is defined as a set of both links and vertices. A further explanation of how this affects the entanglement entropy definition and comparison with other results, especially with the 'edge modes' terms which come entirely from boundaries cutting through links, is a potential area of study. Once the inside links and vertices have been specified, the tracing is done explicitly as an integral over all outside gauge fields and matter fields:

$$\rho(\phi_{\text{in}}, U_{\text{in}}; \phi_{\text{in}}, V_{\text{in}}) = \int [d\phi^4 \dots d\phi^7] \int [dW^{34} \dots dW^{67}] \delta(\Phi(\phi_{\text{in}}, \tilde{\phi}_{\text{out}}; U_{\text{in}}, W_{\text{out}}) \Phi^*(\phi_{\text{in}}, \tilde{\phi}_{\text{out}}; V_{\text{in}}, W_{\text{out}})) \tag{182}$$

The key result is that while term (1) and (2) of Eq. (135) are present in various test meson states at first and second order, term (3) requires going to the third order of perturbative expansions. As term (3) is the distillable entanglement, which can be used as a resource and has the usual interpretation of number of Bell pairs which can be distilled from the state, the results suggest that it is worthwhile to go beyond first-order perturbative expansions.

The calculation shows that even very simple toy models of lattice gauge theories with matter may contain distillable entanglement entropy. This is interesting, especially in the context of studying confinement. So far, e.g. in [29], entanglement entropies of certain QCD systems have been computed numerically using the replica method, which corresponds to treating the state as a member of the whole non-gauge-invariant Hilbert space and calculating the combined entanglement entropy, i.e. all 3 terms together. It could be interesting to extract the distillable part of the entropy in lattice QCD.

8 Conclusion

8.1 Is all entanglement entropy equal?

All the methods presented in this essay seem to calculate entanglement entropy, but are they *guaranteed* to always give the same answer? Let us summarise what we know. We have described and used the following methods:

1. The 'naive' replica trick, often used in continuum methods. It calculates the von Neumann entropy by taking the $n \to 1$ limit of the Renyi entropy. Usually, the method discretises space but not time, and the n-th Renyi entropy is calculated using a Euclidean path integral on n copies of the spatial manifold, glued together at a branch cut at a particular time. Integrating over this n-sheeted manifold is equivalent to tracing over all field configurations, including the non-gauge-invariant states. As shown in [13], it agrees with the GST method and calculates the full entanglement entropy.

   A potential replacement for the replica method, which still uses path integrals and is appropriate for applying in the continuum, is the Euclidean hourglass approach [1]. By changing the way the sheets are glued together, the method cannot produce negative values. For $U(1)$, it has been shown to agree with the *distillable* entanglement, i.e. term (3) in Eq. (183). Both properties are desirable for a potential entanglement measure.

2. The CSH method [8], or algebra of observables method. It involves only gauge-invariant operators. However, it suffers from ambiguities in dividing the links into inside and outside,

as the boundary links can be treated as either inside or outside without changing the defined inside region A. This may lead to gauge-invariant algebras with different centres, and thus - to different superselection sectors and entanglement entropy. Two common choices are the electric centre (maximal choice) and the magnetic centre (minimal choice). The electric centre choice agrees with the GST method.

3. The GST method [13], or maximal extended Hilbert space method. It embeds the gauge-invariant state into a non-gauge-invariant Hilbert space, which is a tensor product of individual link spaces. It agrees with the electric centre construction in the algebra of observables method. An implicit choice was made during the calculations - we worked in the spin eigenbasis when describing the sectors. Different eigenbases may give different entanglement entropies. The agreement with CSH happens when the chosen basis in GST diagonalises all operators in the centre of the chosen CSH algebra. The spin basis is also called the electric basis in this context.

4. The edge states method, developed by Buividovich and Polikarpov [5] and further by Donnely [10, 11]. It is a different kind of extended lattice construction - it defines a region as set of vertices, and chooses an extended Hilbert space with minimal dimension by splitting only the boundary links into two and inserting vertices at the boundary. The resulting lattice is gauge invariant under gauge transformations on all the old vertices, but not on the boundary ones. It has been shown in [8] to agree with CSH with the electric centre choice, and in [13] to agree with GST in the electric basis (i.e. spin basis) choice. However, for the results to agree one must apply the CSH or GST method to the new lattice, i.e. take the 'inside' links to be all old inside links and the 'inside half' of all cut-through links.

In general, the methods agree once we agree to work with the algebra of observables with the maximal centre, the **electric centre**. Studying the entanglement entropy of non-maximal algebras is an interesting avenue to pursue, especially when considering dualities [19]. An example question would be: when non-maximal algebras are mapped using dualities, what are they mapped to? Additionally, in [19] Lin noticed that non-maximal algebras can also be thought of as random boundary conditions at the entangling surface. This has not yet been fully explored.

8.2 Decomposition of entanglement entropy and its interpretation

Finally, we could ask about the physical interpretation of the obtained entanglement entropy terms, hence identifying two possible research directions.

In Eq. (135), there is a clear decomposition of the entanglement entropy:

$$\begin{aligned} S(\rho_A) &= -\sum_{R_\partial} p(R_\partial) \log p(R_\partial) - \sum_{R_\partial} p(R_\partial) \sum_{l \in L_{\partial A}} \log \dim(r_l) - \sum_{R_\partial} p(R_\partial)\, S(\tilde{\rho}_A(R_\partial)) \\ &= \quad \text{(1) classical Shannon entropy of sector probability distribution} \\ &\quad + \text{(2) edge modes (also: } \textit{colour}\text{) entropy} \\ &\quad + \text{(3) quantum von Neumann entropy of each sector (weighted by sector probability)}, \end{aligned} \tag{183}$$

where the sectors here are the gauge-invariant superselection sectors, i.e. within a single sector, one can turn any state into any other with gauge-invariant operators only.

Firstly, are all the terms equally physically relevant? Notice that only (3) looks like a quantum contribution. Also, states from different sectors are unreachable from one another by using gauge-invariant operators only - this was noticed already in Section 7.2.

This bears a contradiction with the **operational view** of entanglement. Originally, in non-gauge lattice systems, the entanglement entropy was proportional to the number of Bell pairs which could be **distilled** from the protocol. That means, it answered the question: how many Bell pairs can we produce from the state using only local operations and classical communication (LOCC)? This definition is sometimes called the **entanglement of distillation**, and is prominent in quantum information theory.

Here is a paradox: in gauge theories, one perhaps *could* go from one state to another using only LOCC, but the required operations could be non-gauge-invariant! If we further require the condition of gauge-invariance, the defining question for entaglement of distillation becomes: how many Bell pairs can we produce from the state using only *gauge-invariant* local operations and classical communication?

As shown in [27] and [30], the terms (1) and (2) are indeed undistillable, and only (3) corresponds to the distillable entanglement entropy. As argued in [30], only (3) is defined with respect to observables, which poses the question - how do terms (1) and (2) manifest physically? Can they be used e.g. as order parameters of phase transitions? Or are all interesting features of full entanglement entropy, e.g. its area laws and non-analytic behaviour in phase transitions, actually a consequence of (3) only? As most studies have been conducted using the full entanglement entropy, this is not fully understood yet.

Secondly, are there any other quantities unique to gauge theories which could be used in parallel to entanglement entropy? Above we focused on distilling Bell pairs within a single sector. But one can ask: is hopping *between* superselection sectors possible, and useful? Despite the problems in turning pure states from one sector into pure states from another, one can still have a superposition of states from many different superselection sectors - in fact, this is what usually happens in the weak coupling regime. One can quantify this statement and introduce a resource called **superselection-induced variance** (SiV)[26], which allows to quantify the degree of 'sector mixing' present in the state. The original definition of SiV is:

$$V(\psi) = 4\left[\langle\psi|\,\hat{N}_A^2\,|\psi\rangle - \langle\psi|\,\hat{N}_A^2\,|\psi\rangle^2\right], \tag{184}$$

where ψ is a state shared between regions A and $\bar{A}$, and $\hat{N}_A$ is the 'particle number operator', whose different eigenvalues divide the Hilbert space into different superselection sectors.

This has been originally found for superselection rules which do not come from gauge theories, but [26] only assumes that the Hilbert space decomposes into a direct sum of superselection sectors, as it does in lattice gauge theories. Hence, it would be interesting to investigate the following similar quantity:

$$V_{ginv}(\psi) = \sum_{i=1}^{k}\left[\langle\psi|\,\mathcal{G}_i^2\,|\psi\rangle - \langle\psi|\,\mathcal{G}_i^2\,|\psi\rangle^2\right], \tag{185}$$

where $\mathcal{G}_i$ are the generators of the centre of the gague-invariant algebra seen in Section 4, whose eigenvalues define the different sectors in lattice gauge theories. This could be a worthwhile quantity

to study, especially when probing the weak coupling regime.

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