UNIVERSITY OF CAMBRIDGE

Faculty of Mathematics

Part III Essay

Chern-Simons Theory and Three-Dimensional Gravity

Guillermo Mera Álvarez

Set by:

Dr. Alejandra Castro

Cambridge

MMXXIV

Contents

Conventions and notation

Units. Default units are $\hbar = c = 1$, but I keep Newton's gravitational constant G explicit.

General relativity. Spacetime will be represented as an (orientable) Lorentzian manifold (M, g). We are interested in three-dimensional gravity, so in principle $\dim M = 3$. I will be using the 'mostly pluses' signature convention for the metric tensor g, which is the most common in the 3d gravity literature. In this convention, the Minkowski metric is $\eta_{ab} = \mathrm{diag}(-1, +1, +1)$. The Einstein-Hilbert action will play a fundamental role in later discussions, so let me define it now as

$$I_{EH}[g] = \frac{1}{16\pi G} \int_M \sqrt{-g}\,(R - 2\Lambda) \tag{0.1}$$

(in Lorentzian signature).

Indices. I use letters from the Greek alphabet $(\mu, \nu, \rho, \ldots)$ for indices that are raised and lowered with the spacetime metric g, and letters from the Latin alphabet $(a, b, c, \ldots)$ for indices that are raised and lowered with the Minkowski metric η. Observe that this notation differs from the one used in Witten's original paper on Chern-Simons theory and 3d gravity [14]. Einstein summation convention is used throughout unless explicitly stated otherwise.

Riemann tensor. I have adopted the following definition for the (components of the) Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma}\,. \tag{0.2}$$

Exterior calculus. If $\{f^\mu\}$ is a basis for T^*M, a p-form $X \in \Omega^p M$ can be written generically as

$$X = \frac{1}{p!} X_{\mu_1 \cdots \mu_p} f^{\mu_1} \wedge \cdots \wedge f^{\mu_p}\,. \tag{0.3}$$

I will use the following convention for the exterior product of a p-form $X \in \Omega^p M$ and a q-form $Y \in \Omega^q M$:

$$(X \wedge Y)_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} = \frac{(p+q)!}{p!\,q!} X_{[\mu_1 \cdots \mu_p} Y_{\nu_1 \cdots \nu_q]}\,. \tag{0.4}$$

And the following definition for the exterior derivative of $X \in \Omega_p M$:

$$(dX)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\,\partial_{[\mu_1} X_{\mu_2 \cdots \mu_{p+1}]}\,. \tag{0.5}$$

The components of the volume form, denoted in this essay by $\varepsilon_{\mu\nu\sigma}$, should not to be confused with the Levi-Civita symbol $\epsilon_{\mu\nu\sigma}$ (the latter is not a tensor in curved space!). Recall that the relationship between these two objects is $\varepsilon_{\mu\nu\sigma} = \sqrt{-g}\,\epsilon_{\mu\nu\sigma}$, which means that, when the metric in consideration is the flat Minkowski metric, both objects are equal: $\varepsilon_{abc} = \epsilon_{abc}$. A general volume form satisfies the following contraction identity:

$$\varepsilon^{\mu_1\cdots\mu_p\kappa_{p+1}\cdots\kappa_n}\varepsilon_{\nu_1\cdots\nu_p\kappa_{p+1}\cdots\kappa_n} = \pm p!\,(n-p)!\,\delta^{\mu_1}_{[\nu_1}\cdots\delta^{\mu_p}_{\nu_p]} \quad \text{for all } p \in \{0,1,\ldots,n\}\,, \tag{0.6}$$

where n is the dimension of the manifold and the plus/minus sign option depends on whether the manifold is Riemannian (plus) or Lorentzian (minus).

Lie algebras. The Lie algebra of a Lie group G will be denoted by either $\mathrm{Lie}(G)$ or $\mathfrak{g}$.

1 Introduction

Perturbative quantum field theory is one of the most capable tools of theoretical physics. Its finest rendition, the Standard Model, gives us almighty predictive power over electromagnetic, weak and strong phenomena. There is, however, one major physical theory that we have not been able to reconcile with quantum field theory: general relativity. General relativity introduces a new, geometric conception of spacetime that radically breaks with the old Newtonian notion of absolute time and space. In this new paradigm, spacetime is a dynamical object itself, and should thus be quantized. But, in breaking with the Newtonian absolutism, general relativity is also distancing itself from some of the basic postulates of quantum field theory. Locality, causality, unitarity; all of these are on the line if we renounce to a Newtonian —or, better said, Lorentzian— conception of spacetime. Therefore, it should not surprise us that, more than 100 years later, the task of finding a consistent quantum theory of gravity remains one of the outstanding problems in theoretical physics. [2]

On top of the conceptual issues, general relativity is a complicated nonlinear theory that offers an all-round resistance to computation. It is natural, then, to look for models that, while mathematically and physically simpler, retain as many quirks of the real problem as possible. One such model is general relativity in 2+1 ("three") dimensions, two spatial and one temporal. As we will argue, three-dimensional general relativity is a theory without propagating metric degrees of freedom, so without gravitational waves. This has two clear advantages with respect to the (3+1)-dimensional case. The first is just analytical convenience. But the second is much more interesting: without propagating degrees of freedom, the role of spacetime topology becomes much clearer.

As first noticed by Achúcarro and Townsend [1], and subsequently developed by Witten [14], the topological nature of three-dimensional general relativity can be made explicit by rewriting the theory as a Chern-Simons theory. Chern-Simons theories are a family of topological gauge theories with an extensively studied quantization, so this formulation opens up a promising pathway towards the quantization of gravity. The objective of this essay is to explore the relationship between three-dimensional gravity and Chern-Simons theory, first at the classical level —in section 2— and then in the context of dS_3 quantum gravity —in section 3.

1.1 Overview

After a case for the study of gravity in three dimensions (sec. 2.1), and short reviews on Chern-Simons theory (sec. 2.2) and the first-order formalism of gravity (sec. 2.3), the Chern-Simons formulation of three-dimensional general relativity is presented in section 2.4. The differences between including a cosmological constant and setting it to zero are large enough to warrant a separate subsection for each case. Both subsections will discuss the explicit construction of the Chern-Simons theory, show the equivalence to the Einstein-Hilbert action, and relate the equations of motion and symmetries of the Chern-Simons theory to those of the metric formulation.

The second part will begin with an introduction to dS_3 quantum gravity (sec. 3.1), where I will also explain the advantages of working with a positive cosmological constant. The objective will be to test the equivalence between the metric and Chern-Simons formulations at tree and one-loop level in perturbation theory. The tree-level (sec. 3.2) and one-loop (sec. 3.3) contributions to the partition function will be computed in the metric formulation and then compared to Chern-Simons results in the literature.

2 A topological perspective on classical 3d gravity

The goal of this first part of the essay is to formulate three-dimensional (3d) general relativity as a Chern-Simons gauge theory, which is a topological theory. The motivation behind this programme will be outlined in section 2.1, where I will discuss the significance of topology in a gravitational context and justify the choice of dimension. The next two sections will quickly review the Chern-Simons action —sec. 2.2— and the first-order formalism for 3d gravity —sec. 2.3. Readers familiar with this material can skip to section 2.4, which is where the Chern-Simons formulation of 3d gravity —with and without cosmological constant— will be described in full detail.

2.1 Motivation

The object of study of a theory of gravity is spacetime itself, represented as a metric manifold (M, g). The standard introduction to gravitation focuses almost entirely on describing how Einstein's equations constrain the local metric structure of this manifold. This is of course a major step in understanding the dynamics of spacetime, but says nothing about the other defining ingredient of a metric manifold: its topology. Topological features are known to produce measurable effects in other physical settings —think, for instance, of the Aharonov-Bohm effect—, hence it seems natural to wonder if this is also the case in the context of general relativity. If the answer is yes, then there is a clear quantum-level implication: the topology of spacetime may also be quantum. In particular, a putative quantum theory of gravity should at the very least allow for different topologies to contribute to the path integral.

In dimension three (2+1) and lower (1+1), the metric degrees of freedom of classical gravity do not propagate, which clears the view for identification of potential topological effects. (2+1)-dimensional gravity exhibits this interesting characteristic while still retaining much of the conceptual richness of (3+1)-gravity. This makes (2+1)-dimensional gravity the perfect theoretical laboratory to test our ideas on the topology of spacetime.

To see why there are no propagating gravitational degrees of freedom in dimension three, start by considering the more general case of an n-dimensional spacetime (M, g). At first sight, the Riemann tensor has a total of n^4 components; $R_{\mu\nu\rho\sigma}$ with μ, ν, ρ, σ taking values from 0 to $n-1$. But the symmetries of this object establish relationships between these components, meaning that not all of them are actually independent [7]:

- Antisymmetry in the first pair of indices means that there are only $N = \frac{n(n-1)}{2}$ independent choices of (μ, ν). The same applies to the second pair of indices, (ρ, σ).

- Symmetry under the exchange of both pairs of indices reduces the number of independent choices to

$$\left(\!\!\binom{N}{2}\!\!\right) = \binom{2+N-1}{2} = \frac{(N+1)N}{2}\,. \tag{2.1}$$

  Here $\left(\!\binom{n}{k}\!\right) := \binom{k+(n-1)}{k}$ denotes the number of k-combinations with repetition that one can extract from a set with n elements.

- A further constraint is given by the algebraic (or first) Bianchi identity:

$$R_{\mu[\nu\rho\sigma]} = 0 \iff R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0\,. \tag{2.2}$$

  If any two indices are equal, this identity reduces to the previous symmetry and anti-symmetry relations between components. In all the other $\binom{n}{k}$ cases, the identity strips the Riemann tensor of one degree of freedom.

So the actual number of independent components of the Riemann tensor in n dimensions is

$$\frac{1}{2}\binom{n}{2}\left[\binom{n}{2}+1\right] = \frac{1}{12}\,n^2(n^2-1)\,. \tag{2.3}$$

Let me introduce now the Weyl tensor, which should help us to see why the $n=3$ case is special in some sense. This object is what remains of the Riemann tensor after removing all its contractions [3]:

$$W_{\mu\nu\rho\sigma} \coloneqq R_{\mu\nu\rho\sigma} - \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \frac{2}{(n-1)(n-2)}\,g_{\mu[\rho}g_{\sigma]\nu}R\,. \tag{2.4}$$

The Weyl tensor has all the symmetries of the Riemann tensor,

- $W_{[\mu\nu][\rho\sigma]} = W_{\mu\nu\rho\sigma}$,
- $W_{\mu\nu\rho\sigma} = W_{\rho\sigma\mu\nu}$,
- $W_{\mu\nu\rho\sigma} + W_{\mu\sigma\nu\rho} + W_{\mu\rho\sigma\nu} = 0$,

so it has the same number of independent components as $R_{\mu\nu\rho\sigma}$. But it has also been defined so as to satisfy the following $\frac{n(n+1)}{2}$ equations:

$$W^{\mu}{}_{\nu\mu\sigma} = 0\,. \tag{2.5}$$

For $n=3$, the number of independent components of the Weyl tensor happens to be equal to the number of additional constraints imposed by the previous equation,

$$\frac{1}{12}n^2(n^2-1) = 6 = \frac{1}{2}n(n+1)\,. \tag{2.6}$$

As a result, the Weyl tensor is identically zero. Looking back at (2.4), this means that the three-dimensional Riemann tensor is fully determined in terms of the Ricci tensor, the Ricci scalar and the metric:

$$R_{\mu\nu\rho\sigma} = 2(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - g_{\mu[\rho}g_{\sigma]\nu}R\,. \tag{2.7}$$

Now recall the form of the vacuum Einstein equations (VEE) with cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{2.8}$$

When these equations hold, they determine the Ricci tensor and Ricci scalar as functions of the metric components, regardless of the dimension of our manifold. Indeed, taking the trace of (2.8) gives $R = \frac{2n}{n-2}\Lambda$, which fixes the Ricci scalar. Substituting back into the VEE, we arrive at $R_{\mu\nu} = \frac{2}{n-2}\Lambda g_{\mu\nu}$, which for $n = 3$ takes the form

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \,. \tag{2.9}$$

Then equation (2.7) becomes

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}) \tag{2.10}$$

Notice that no derivatives of the metric appear in the on-shell Riemann tensor. In classical three-dimensional gravity, the curvature of spacetime at a given point only depends on the components of the metric at such point. In this scenario, the metric degrees of freedom of two neighbouring points cannot interact, and hence these degrees of freedom cannot propagate. In summary, there are no gravitational waves in three dimensions.

2.2 The Chern-Simons action

The absence of propagating degrees of freedom might even mean that classical 3d gravity is a topological theory, this is, a theory whose classical phase space —the space of solutions to the equations of motion, modulo gauge transformations— only depends on the topology of spacetime. We will make this explicit in section 2.4 by showing that the Einstein-Hilbert action is equivalent to the action of a well-known topological field theory: Chern-Simons theory. But, for now, let's review the main features of classical Chern-Simons theory [2].

Let (M, g) be an oriented spacetime manifold of dimension three and G be a compact, simple Lie group[1]. Choosing a basis $\{T^a\}$ for Lie(G), let

$$A = A_\mu(x)dx^\mu \,, \quad A_\mu(x) = A^a_\mu(x)T_a \in \mathrm{Lie}(G) \tag{2.11}$$

define the vector potential for a gauge theory with gauge group G. (Mathematically, A is a connection 1-form on a principal G-bundle.) The *Chern-Simons action* for A is

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \,. \tag{2.12}$$

Here k is a coupling constant, called the *level* of the theory, and Tr is a non-degenerate, invariant[2] bilinear form on Lie(G) (e.g., the matrix trace in a suitable representation). In spite of its rather awkward form, the Chern-Simons action is of course invariant under gauge

[1] Some of the hypotheses imposed on G can be relaxed.

[2] Let V be a representation of a Lie algebra $\mathfrak{g}$ over a field K. A bilinear form $B\colon V \times V \to K$ is called *invariant* if

$$B(Xv, w) + B(v, Xw) = 0$$

for all $X \in \mathfrak{g}$ and $v, w \in V$. In this essay, invariant form are always defined over $V = \mathfrak{g} = \mathrm{Lie}(G)$ with the adjoint representation.

transformations. For some purposes it could be useful to have a component-explicit form of the action. Say $[T_a, T_b] = f_{ab}^c T_c$ and $\kappa_{ab} := \mathrm{Tr}(T_a T_b)$, then[3]

$$S_{\mathrm{CS}}[A] = -\frac{k}{4\pi}\int_M d^3x\sqrt{-g}\,\kappa_{ab}\,\varepsilon^{\rho\lambda\kappa}\left(A^a_\rho\partial_\lambda A^b_\kappa + \frac{1}{3}A^a_\rho f^b_{cd}A^c_\lambda A^d_\kappa\right). \tag{2.13}$$

What are the classical equations of motion for the action (2.12)? Under an infinitesimal variation $A \to A + \delta A$, the change in the action is [11]

$$\delta S_{\mathrm{CS}}[A,\,\delta A] = \frac{k}{4\pi}\left\{\int_M d\,\mathrm{Tr}(A\wedge\delta A) + 2\int_M \mathrm{Tr}(\delta A\wedge F[A])\right\} \tag{2.14}$$

where $F[A] := dA + A\wedge A$ is the field strength (curvature form) of A. Applying Stokes' theorem to the first term in this expression gives $\int_{\partial M}\mathrm{Tr}(A\wedge\delta A)$, which vanishes after requiring that $\delta A|_{\partial M} = 0$. The extremal points of the action are then be given by

$$\boxed{F[A] = dA + A\wedge A = 0}\,. \tag{2.15}$$

So the classical solutions of Chern-Simons theory are those field configurations that have a vanishing field strength. Mathematically, these solutions correspond to flat connections on some principal G-bundle P over M.

Let's finish this section by showing that Chern-Simons theory is indeed a topological theory. For gauge group G and spacetime manifold M, equation (2.15) defines the classical phase space of the theory as

$$\mathfrak{F}_{M,G} = \{(P,A)\colon P\to M \text{ principal } G\text{-bundle},\ A \text{ flat connection on } P\}/\sim, \tag{2.16}$$

where $(P,A)\sim(P',A')$ if and only if there exists a isomorphism of principal bundles $\varphi\colon P\to P'$ such that $\varphi^*A' = A$ (this is just a formal way of saying that A and A' are related by a gauge transformation). The relationship between the classical phase space and the topology of M is then provided by the following theorem:

Theorem 2.1 (Classification of flat connections)**.** *The set* $\mathfrak{F}_{M,G}$ *is in bijective correspondence with the set* $\mathrm{Hom}(\pi_1(M);G)/G$.

Here, $\pi_1(M)$ denotes the fundamental group of M, which is a topological property, and $\mathrm{Hom}(\pi_1(M);G)/G$ denotes the set of orbits of $\mathrm{Hom}(\pi_1(M),G)$ under the action

$$\begin{aligned} G\times\mathrm{Hom}(\pi_1(M);G) &\longrightarrow \mathrm{Hom}(\pi_1(M);G)\\ (g,\rho) &\longmapsto \rho'(\cdot) = g\rho(\cdot)g^{-1}\,. \end{aligned} \tag{2.17}$$

I will not give a full proof of the theorem, the interested reader can find it in [12, §13.9]. It will be enough for us to know that the correspondence works in the following way: each flat

[3] The structure constants come from $(A\wedge A)_{\mu\nu} = 2A_{[\mu}A_{\nu]} = [A_\mu, A_\nu]$ (recall that we are working with Lie algebra-valued 1-forms).

connection $[(P, A)]$ is mapped to the homomorphism $[\rho_A] \in \mathrm{Hom}(\pi_1(M); G)/G$ that takes a loop $[\gamma] \in \pi_1(M)$ with basepoint x to its holonomy at any chosen p in the fibre of x, $h_{A,\gamma}(p)$. This correspondence shows that the space of solutions of Chern-Simons theory is fully determined by the topology of M, hence the theory is topological.

Remark. The careful reader will have noticed that I have silently assumed the gauge invariance of the Chern-Simons action, which is not obvious from the form of the action. A possible proof would involve showing that, if Y is a 4-manifold with boundary M, the —manifestly gauge-invariant— action $\int_Y \mathrm{Tr}(F \wedge F)$ is a total derivative that reduces to the Chern-Simons action on M [14]. But this computation is not too relevant for our purposes, so I have chosen to leave it out.

2.3 The first-order formalism

The relationship with Chern-Simons theory does not come directly from the standard form of the Einstein-Hilbert action. We have to first shift to what is often known as the *first-order formulation of gravity*, as opposed to the more conventional metric —"second-order"— formulation. This section mainly follows [2, §2.5], with some material coming from [7, §2.4] as well.

2.3.1 The Palatini action

Let's start by introducing the two objects that will act as fundamental variables when we move to the new formalism: the triad and the spin connection. A *triad* or *dreibein* is a section $\{e_a = e_a^\mu \partial_\mu\}$ of the orthonormal frame bundle of TM, the tangent bundle to M. The orthonormality condition is

$$g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab} \tag{2.18}$$

The dreibein is an orthonormal basis of vector fields over M, and so it has a dual basis, given by $e_\mu^a := \eta^{ac} g_{\mu\nu} e_c^\nu$. By definition, $e_\mu^a e_b^\mu = \delta_b^a$. In addition, the dual basis satisfies the following identity:

$$\eta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu} \,, \tag{2.19}$$

or, equivalently, $e_\mu^a e_a^\nu = \delta_\mu^\nu$. These properties mean that we can effectively raise and lower Greek indices with $g_{\mu\nu}$ and raise and lower Latin indices with η_{ab}. The *spin connection* ${\omega^a}_{b\mu}$ is defined as a collection of *connection 1-forms*

$$({\omega^a}_b)_\mu := e_\nu^a \nabla_\mu e_b^\nu = \Gamma_{bc}^a e_\mu^c \,. \tag{2.20}$$

The most important property of the spin connection is antisymmetry in its Minkowski indices, $(\omega_{ab})_\mu = -(\omega_{ba})_\mu$. The dreibein and the spin connection also satisfy the so-called *Cartan* *structure equations*:

$$de^a + \omega^a{}_b \wedge e^b = 0\,, \tag{2.21}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \Theta^a{}_b\,, \tag{2.22}$$

where $\Theta^a{}_b = \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d$ are the *curvature 2-forms*. The curvature 2-forms encode the components of the Riemann tensor with respect to the dreibein:

$$e^\mu_c e^\nu_d (\Theta^a{}_b)_{\mu\nu} = R^a{}_{bcd}\,. \tag{2.23}$$

Remember that the objective of this part of the essay is to get a gauge theory out of the three-dimensional Einstein-Hilbert action. In the gauge theory formulation, the Minkowski indices associated to the frame fields will become component indices of the Lie algebra of the gauge group. Formally, a gauge field is a Lie algebra-valued 1-form, so we should seek for 1-forms that have only one Minkowski index. Obtaining an object of this kind from the dreibein is of course an easy task: one can always consider the dual field $e^a = e^a_\mu dx^\mu$. Ideally, we would like to do something similar with the spin connection. In three spacetime dimensions, this is possible thanks to the existence of the completely antisymmetric tensor ε^{abc}:

$$\omega^a := \frac{1}{2}\varepsilon^{abc}(\omega_{bc})_\mu dx^\mu = \frac{1}{2}\varepsilon^{abc}\omega_{bc} \iff \omega_{ab} = -\varepsilon_{abc}\omega^c\,. \tag{2.24}$$

With a similar trick, and making use of the second Cartan structure equation (2.22),

$$\Theta^a := \frac{1}{2}\varepsilon^{abc}\Theta_{bc} = d\omega^a + \frac{1}{2}\varepsilon^a{}_{bc}\omega^b \wedge \omega^c\,, \tag{2.25}$$

The idea behind the first order formalism is to stop using the metric as the dynamic variable of the theory and start using the triad and the (modified) spin connection ω^a instead. From this new perspective, the metric is just the function of the dreibein given by equation (2.19). In terms of the new variables, the Einstein-Hilbert action (0.1) takes the form

$$\boxed{I_{\text{EH}}[e,\omega] = \frac{1}{8\pi G}\int_M \left\{ e^a \wedge \left(d\omega_a + \frac{1}{2}\varepsilon_a{}^{bc}\omega_b \wedge \omega_c \right) - \frac{\Lambda}{6}\varepsilon_{abc}e^a \wedge e^b \wedge e^c \right\}}\,, \tag{2.26}$$

sometimes called the *Palatini action*. To reproduce this result, one has to show that

$$2\int_M e^a \wedge \Theta_a \equiv \int_M \varepsilon_{abc}e^a \wedge \Theta^{bc} = \int_M d^3x\sqrt{-g}R\,. \tag{2.27}$$

and

$$\int_M \varepsilon_{abc}e^a \wedge e^b \wedge e^c = 3!\int_M d^3x\sqrt{-g}\,, \tag{2.28}$$

which is straightforward once in possession of the following bits of information:

1. Equation (2.19).

2. Equation (2.23).

3. $\varepsilon_{abc} \equiv \varepsilon(e_a, e_b, e_c) = e^\mu_a e^\nu_b e^\kappa_c \varepsilon(\partial_\mu, \partial_\nu, \partial_\kappa) = e^\mu_a e^\nu_b e^\kappa_c \varepsilon_{\mu\nu\kappa}$.

4. Equation (0.6), case $n = 3$.

2.3.2 Equations of motion

The next step of the analysis is extracting the equations of motion associated to this action. On the one hand, the equation coming from the variation of ω is

$$T_a := \boxed{de_a + \varepsilon_{abc}\omega^b \wedge e^c = 0} . \tag{2.29}$$

This is the "no torsion" condition, which can be interpreted as a compatibility condition between the spin connection and the dreibein. As long as e^a_μ is invertible, the equation can be solved for ω.

On the other hand, varying e leads us to

$$\Theta^a \equiv \boxed{d\omega_a + \frac{1}{2}\varepsilon_{abc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\varepsilon_{abc}e^b \wedge e^c} , \tag{2.30}$$

which is just the condition that spacetime has constant curvature:

$$\Theta_{ab} = -\varepsilon_{abc}\Theta^c = \Lambda e_a \wedge e_b \, . \tag{2.31}$$

Let me stress here that the equivalence of the first- and second-order formalisms is only partial: the first-order field equations only yield a solution of the second-order equations when the triad e^a_μ is invertible. For classical applications, the requirement of invertibility is a natural one, but non-invertible triads could in principle be important in the quantum theory [2].

2.3.3 Symmetries of the first-order Einstein-Hilbert action

Up to boundary terms, the Palatini action is invariant under two sets of local symmetries: *local Lorentz transformations*,

$$\begin{cases} \delta e^a = \varepsilon^{abc} e_b \tau_c \\ \delta\omega^a = d\tau^a + \varepsilon^{abc}\omega_b\tau_c \end{cases} , \tag{2.32}$$

and *local translations*,

$$\begin{cases} \delta e^a = d\rho^a + \varepsilon^{abc}\omega_b\rho_c \\ \delta\omega^a = -\Lambda\varepsilon^{abc}e_b\rho_c \end{cases} . \tag{2.33}$$

These symmetries can be traced back to the familiar diffeomorphism invariance of the Einstein-Hilbert action. Let's see how.

By definition, the change of a general tensor field under the infinitesimal diffeomorphism generated by a vector field ξ is given by its Lie derivative with respect to ξ. When the tensor field is a 1-form, its Lie derivative can be expressed as

$$\mathcal{L}_\xi \sigma = d(i_\xi \sigma) + i_\chi d\sigma \equiv d(\xi \cdot \sigma) + \xi \cdot d\sigma \,, \tag{2.34}$$

Here $i_\xi Z$ denotes the $(q-1)$-form resulting from contracting ξ with the first index of some q-form Z. Applying this reasoning to the 1-forms e^a and ω^a, the change in e and ω under the infinitesimal diffeomorphism generated by ξ is

$$\begin{aligned}\delta e^a = \mathcal{L}_\xi e^a = d(\xi \cdot e^a) + \xi \cdot de^a = d(\xi \cdot e^a) + \varepsilon^{abc}\omega_b(\xi \cdot e_c) + \varepsilon^{abc} e_b(\xi \cdot \omega_c) \\ + \text{(terms proportional to the e.o.m.)} \,,\end{aligned} \tag{2.35}$$

$$\begin{aligned}\delta \omega_a = \mathcal{L}_\xi \omega^a = d(\xi \cdot \omega^a) + \xi \cdot d\omega^a = d(\xi \cdot \omega^a) + \varepsilon^{abc}\omega_b(\xi \cdot \omega_c) - \Lambda \varepsilon^{abc} e_b(\xi \cdot e_c) \\ + \text{(terms proportional to the e.o.m.)} \,.\end{aligned} \tag{2.36}$$

These may be recognized as (2.32) + (2.33) with parameters

$$\rho^a = \xi \cdot e^a \,, \quad \tau^a = \xi \cdot \omega^a \,. \tag{2.37}$$

Hence (2.32) + (2.33) are equivalent, *on-shell*, to infinitesimal diffeomorphisms.

2.4 Chern-Simons formulation of 3d gravity

We are now ready to tackle the Chern-Simons formulation of three-dimensional gravity. Following [14], the cases $\Lambda = 0$ and $\Lambda \neq 0$ are treated separately. In each case I will:

i. provide a explicit construction of the Chern-Simons theory,

ii. show the equivalence of the theories at the action level,

iii. relate their equations of motion, and

iv. relate the gauge transformations of the Chern-Simons theory to the symmetries of the gravitational action.

2.4.1 Zero cosmological constant

Construction of the theory. A Chern-Simons action consists of three primary components: the gauge group G, an invariant and non-degenerate bilinear form on Lie(G), and the gauge potential A.

Let's first specify the gauge group of the theory. In the absence of a cosmological constant, the 3d Einstein's equations prescribe the flatness of spacetime[4]. The simply connected covering base of such a spacetime is Minkowski space, which has isometry group ISO$(1,2)$. The dreibein expresses this covering by mapping the tangent coordinate basis at each point to the standard vector basis of Minkowski space; hence we can expect some sort of ISO$(1,2)$

[4] In three spacetime dimensions, $R_{\mu\nu} = 0$ (Ricci flatness) $\implies R_{\mu\nu\rho\sigma} = 0$ (flatness); see section 2.1.

symmetry from this object. Having already stated our intention of building the gauge potential out of the dreibein (and the spin connection), it seems that basing our theory on $G = \mathrm{ISO}(1,2)$ would be sensible.

Next, we need to make sure that $\mathfrak{iso}(1,2) \equiv \mathrm{Lie}(\mathrm{ISO}(1,2))$ admits an invariant and non-degenerate bilinear form. Let J^{ab} denote the generators of Lorentz transformations and P^a denote the generators of translations. These obey

$$\begin{aligned} &[J^{ab}, J^{cd}] = \eta^{ac} J^{bd} - \eta^{ad} J^{bc} - \eta^{bc} J^{ad} - \eta^{bc} J^{ad} + \eta^{bd} J^{ac} \,, \\ &[J^{ab}, P^c] = \eta^{ac} P^b - \eta^{bc} P^a \,, \qquad [P^a, P^b] = 0 \,. \end{aligned} \tag{2.38}$$

In three spacetime dimensions, J^{ab} can be replaced by the more convenient $J^a = \frac{1}{2}\epsilon^{abc} J_{bc}$. The commutation relations then become

$$\begin{aligned} &[J^a, J^b] = \epsilon^{abc} J_c \,, \\ &[J^a, P^b] = \epsilon^{abc} P_c \,, \\ &[P^a, P^b] = 0 \,. \end{aligned} \tag{2.39}$$

We can define an invariant and non-degenerate bilinear form on this algebra by

$$\left.\begin{aligned} \kappa(J^a, P^b) = \eta^{ab} = \kappa(P^b, J^a) \\ \kappa(J^a, J^b) = \kappa(P^a, P^b) = 0 \end{aligned}\right\} \tag{2.40}$$

(checking that κ has the desired properties is straightforward). From now on, we will denote

$$\kappa(X, Y) \equiv \mathrm{Tr}(XY) \,. \tag{2.41}$$

The last ingredient of our theory is the gauge potential, which we will define to be

$$A_\mu = e^a_\mu P_a + \omega^a_\mu J_a \;\in\; \mathfrak{iso}(1,2)) \,. \tag{2.42}$$

We can think of A as belonging to the adjoint representation of $\mathrm{ISO}(1,2)$, as expected for a gauge field.

Equivalence at action level. Recall the form of the Chern-Simons action:

$$\begin{aligned} S_{\mathrm{CS}}[A] &= \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \\ &= -\frac{k}{4\pi} \int_M \sqrt{-g}\, \varepsilon^{\rho\lambda\kappa}\, \mathrm{Tr}\left(A_\rho \partial_\lambda A_\kappa + \frac{1}{3} A_\rho [A_\lambda, A_\kappa] \right) \,. \end{aligned} \tag{2.43}$$

For the gauge field defined in (2.42), the commutator in this expression is

$$[A_\lambda, A_\kappa] = \epsilon_{abc} (e^a_\lambda \omega^b_\kappa - e^a_\kappa \omega^b_\lambda) P^c + \epsilon_{abc} \omega^a_\lambda \omega^b_\kappa J^c \,. \tag{2.44}$$

Substituting this and (2.42) into (2.43) and applying the definition of Tr gives

$$S_{\text{CS}}[A] = -\frac{k}{4\pi}\int_M d^3x\sqrt{-g}\,\varepsilon^{\rho\lambda\kappa}e^a_\rho(\partial_\lambda\omega_{a\kappa} - \partial_\kappa\omega_{a\lambda} + \epsilon_{abc}\omega^b_\lambda\omega^c_\kappa) \\ -\frac{k}{4\pi}\int_M d^3x\sqrt{-g}\,\varepsilon^{\rho\lambda\kappa}(\omega_{a\lambda}\partial_\kappa e^a_\rho + e^a_\rho\partial_\kappa\omega_{a\lambda})\,. \tag{2.45}$$

Meanwhile, setting $\Lambda = 0$ in the first-order Einstein-Hilbert action (2.26) leaves us with

$$I_{EH} = \frac{1}{8\pi G}\int_M e^a \wedge (d\omega_a + \frac{1}{2}\varepsilon_{abc}\omega^b \wedge \omega^c)\,. \tag{2.46}$$

In component form:

$$I_{EH} = -\frac{1}{16\pi G}\int_M d^3x\sqrt{-g}\,\varepsilon^{\rho\lambda\kappa}e^a_\rho(\partial_\lambda\omega_{a\kappa} - \partial_\kappa\omega_{a\lambda} + \varepsilon_{abc}\omega^b_\lambda\omega^c_\kappa) \tag{2.47}$$

This matches the first term in (2.45) after setting $k = \frac{1}{4G}$ and noticing that $\varepsilon_{abc} = \epsilon_{abc}$ (see "Conventions and notation").

If the second term in (2.45) is a boundary term, we are done. Happily for us,

$$\int_{\partial M} e^a \wedge \omega_a = \int_M d(e^a \wedge \omega_a) = -\int_M d^3x\sqrt{-g}\,\varepsilon^{\rho\lambda\kappa}(\omega_{a\lambda}\partial_\kappa e^a_\rho + e^a_\rho\partial_\kappa\omega_{a\lambda})\,, \tag{2.48}$$

ergo, up to boundary terms, the Chern-Simons action that we have constructed is equivalent to the $\Lambda = 0$ Einstein-Hilbert action.

Remark. The argument above relies on the equivalence of the first- and second-order formalisms of gravity, which is only valid under the assumption of an invertible dreibein.

Relationship between equations of motion. Substituting $A = e^a P_a + \omega^a J_a$ into the equations of motion of the Chern-Simons action gives

$$0 = (dA + A \wedge A)_{\mu\nu} = \Big((de_c)_{\mu\nu} + \varepsilon_{cab}(e^a_\mu\omega^b_\nu - e^a_\nu\omega^b_\mu)\Big)\,P^c \\ + \Big((d\omega_c)_{\mu\nu} + \frac{1}{2}\varepsilon_{cab}(\omega^a_\mu\omega^b_\nu - \omega^a_\nu\omega^b_\mu)\Big)\,J^c\,. \tag{2.49}$$

The generators $\{P_c, J_c\}$ form a basis for the Lie algebra, so this is equivalent to

$$\left.\begin{aligned} de_a + \varepsilon_{abc}\omega^b \wedge e^c = 0 \\ d\omega_a + \tfrac{1}{2}\varepsilon_{abc}\omega^a \wedge \omega^b = 0 \end{aligned}\right\}\,, \tag{2.50}$$

which —as seen in (2.29)–(2.30)— are the equations of motion of the $\Lambda = 0$ Einstein-Hilbert action in the first order formalism.

Relationship between symmetries. A natural question at this stage is how to interpret the ISO(1, 2) gauge symmetry from the point of view of the gravitational theory. Is it related in any way to the diffeomorphism invariance of the Einstein-Hilbert action?

To answer these questions, consider an infinitesimal gauge transformation $\Omega(x) = e^{u(x)} \approx \mathbf{1} + u(x) + \dots$ with

$$u(x) = \rho^a(x) P_a + \tau^a(x) J_a \,, \quad |\rho^a|, |\tau^a| \ll 1 \,. \tag{2.51}$$

Gauge invariance requires that the variation of A_μ under such transformation is

$$\delta A_\mu = \mathcal{D}_\mu u \,, \tag{2.52}$$

where, by definition,

$$\mathcal{D}_\mu u = \nabla_\mu u + [A_\mu, u] = \partial_\mu u + [A_\mu, u] \,. \tag{2.53}$$

Inserting the definitions of A and u into this expression,

$$[A_\mu, u] = \varepsilon_{abc} e^a_\mu \tau^b P^c + \varepsilon_{abc} \omega^a_\mu \rho^b P^c + \varepsilon_{abc} \omega^a_\mu \tau^b J^c \,, \tag{2.54}$$

so we arrive at

$$\left. \begin{array}{l} \delta e^a_\mu = \partial_\mu \rho^a + \varepsilon^{abc} e_{b\mu} \tau_c + \varepsilon^{abc} \omega_{b\mu} \rho_c \\ \delta \omega^a_\mu = \partial_\mu \tau^a + \varepsilon^{abc} \omega_{b\mu} \tau_c \end{array} \right\} \,. \tag{2.55}$$

The above expressions agree with those describing the "local Lorentz transformations" (2.32) + "local translations" (2.33) symmetry of the first-order Einstein-Hilbert action in the case of a zero cosmological constant. Since these transformations are equivalent, on shell, to infinitesimal diffeomorphisms, so are the gauge transformations of our Chern-Simons theory.

This representation of infinitesimal[5] diffeomorphisms as gauge transformations is characteristic of topological field theories. The advantages of replacing the labyrinthine diffeomorphism group with pointwise gauge transformations become obvious, for example, when negotiating the quantization of the theory [2]. The quantization of Chern-Simons theories is, in particular, relatively well understood; hence our commitment to a Chern-Simons formulation of gravity.

2.4.2 Non-zero cosmological constant

We would now like to generalize the construction to include a cosmological constant. In this section I will use $\mathfrak{g}$ to denote the Lie algebra of a Lie group G.

Construction of the theory. For non-zero values of the cosmological constant, solutions to the 3d Einstein's equations are no longer flat, but locally homogeneous spacetimes. The simply connected covering base of such a spacetime is not Minkowski space, but de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) space[6]. The isometry group of these two spaces is not $\mathrm{ISO}(1,2)$, but $\mathrm{SO}(1,3)$ and $\mathrm{SO}(2,2)$, respectively. Drawing on the reasoning from the $\Lambda = 0$

[5] "Large" diffeomorphisms (those that cannot be smoothly deformed to the identity) must be treated separately. Although mostly overlooked in this essay, these symmetries play an important role in the quantum theory.

[6] Cartan geometry might be a more suitable framework to conduct the analysis.

case, we may guess that 3d gravity with a non-zero cosmological constant will be related to Chern-Simons theories of these latter groups.

The Lie algebras of both $\mathrm{SO}(1,3)$ and $\mathrm{SO}(2,2)$ are connected to $\mathfrak{iso}(1,2)$ via the following group contraction[7]:

$$\begin{aligned} &[J_a, J_b] = \epsilon_{abc} J^c\,, \quad [J_a, P_b] = \epsilon_{abc} P^c\,, \\ &[P_a, P_b] = -\Lambda \epsilon_{abc} J^c\,. \end{aligned} \tag{2.56}$$

Observe that, if $\Lambda = \pm 1/l^2$, then

$$\mathrm{span}_{\mathbb{R}}\{J_a, P_a\} = \mathrm{span}_{\mathbb{R}}\left\{J_a, \frac{P_a}{l}\right\} \cong \begin{cases} \mathfrak{so}(1,3) & \text{if } \Lambda > 0 \\ \mathfrak{so}(2,2) & \text{if } \Lambda < 0 \end{cases}\,, \tag{2.57}$$

as advertised above.

We could now repeat what we did for $\Lambda = 0$: introduce a $\mathrm{SO}(1,3)$ or $\mathrm{SO}(2,2)$ gauge potential $A = e^a P_a + \omega^a J_a$, show that the corresponding gauge transformations are of the form (2.32) + (2.33) and, using the bilinear form (2.40), find that the Chern-Simons action of A comes out to be precisely the first-order Einstein-Hilbert action (up to possible boundary terms).

Let's try a different route instead. We start by performing the change of basis

$$J_a^{\pm} := \frac{1}{2}\left(J_a \pm \frac{1}{\sqrt{-\Lambda}} P_a\right)\,, \tag{2.58}$$

(requires $\Lambda \neq 0$). For $\Lambda > 0$, $\sqrt{-\Lambda}$ is complex, which means that we have implicitly moved to $\mathfrak{so}(1,3)_{\mathbb{C}}$. In any case, the commutation relations become simply:

$$\begin{aligned} &[J_a^+, J_b^+] = \epsilon_{abc} J^{+c}\,, \\ &[J_a^-, J_b^-] = \epsilon_{abc} J^{-c}\,, \\ &[J_a^+, J_b^-] = 0\,. \end{aligned} \tag{2.59}$$

These are the commutation relations of two mutually commuting copies of $\mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R})$ ($\mathrm{SL}(2,\mathbb{R})$ is the double cover of $\mathrm{SO}(1,2)$ [9]). For $\Lambda < 0$, this has the following straightforward interpretation:

$$\mathfrak{so}(2,2) = \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})\,, \tag{2.60}$$

where $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ denotes the Lie algebra of the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. For $\Lambda > 0$, the interpretation is a bit more involved. In this case, the splitting (2.59) is

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}\,. \tag{2.61}$$

[7] When comparing to [14], be careful to note that what Witten calls λ corresponds to $-\Lambda$ in my conventions.

Then, noticing that $\mathrm{SL}(2,\mathbb{R})_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2)_{\mathbb{C}}$,

$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} = (\mathfrak{su}(2) \oplus \mathfrak{su}(2))_{\mathbb{C}} = (\mathfrak{su}(2) \times \mathfrak{su}(2))_{\mathbb{C}} \,. \tag{2.62}$$

Now define the connection $A = A^{\pm a} J_a^{\pm}$ with

$$A^{\pm a} = \omega^a \pm \sqrt{-\Lambda}\, e^a \,, \tag{2.63}$$

The above splittings allow for the interpretation of A as two $\mathfrak{g}$-valued connections, $A^{(+)} = A^{+a} L_a$ and $A^{(-)} = A^{-a} L_a$, where

→ $G = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SO}(1,2)$ for $\Lambda < 0$,

→ $G = \mathrm{SU}(2)$ for $\Lambda > 0$,

and $\{L_a\}$ are the images of $J_a^{\pm}$ under the splitting.

This Lie algebra $\mathfrak{g}$ admits an obvious non-degenerate, invariant bilinear form:

$$\kappa(L_a, L_b) = \eta_{ab} \equiv 2\,\mathrm{Tr}(L_a L_b) \,. \tag{2.64}$$

The factor of two has been added for conformity with [7].

Remark. For the rest of this section, I will use the notation

$$e \equiv e^a L_a \,, \quad \omega \equiv \omega^a L_a \,. \tag{2.65}$$

Observe that:

$$\omega \wedge \omega = \frac{1}{2}[L^a, L^b]\, \omega_a \wedge \omega_b = \frac{1}{2}\epsilon^{abc}(\omega_a \wedge \omega_b) L_c \,, \tag{2.66}$$

$$\omega \wedge e + e \wedge \omega = [L^a, L^b](\omega_a \wedge e_b) = \epsilon^{abc}(\omega_a \wedge \omega_b) L_c \,, \tag{2.67}$$

$$\mathrm{Tr}(e \wedge e \wedge e) = \frac{1}{2}\epsilon^{abc}\,\mathrm{Tr}(L_d L_c)(e^d \wedge e_a \wedge e_b) = \frac{1}{2}\epsilon^{abc}(e_a \wedge e_b \wedge e_c) \,. \tag{2.68}$$

Equivalence at action level. Our starting point will be the action

$$S[A^{(+)}, A^{(-)}] \coloneqq S_{\mathrm{CS}}[A^{(+)}] - S_{\mathrm{CS}}[A^{(-)}] \,. \tag{2.69}$$

with the assumption that the levels of the "plus" and "minus" components of the action are equal; $k_+ = k_- \equiv k$. Substituting $A^{(\pm)} = \omega \pm \sqrt{-\Lambda}\, e$ into the right hand side of (2.69) and expanding,

$$S[A^{(+)}, A^{(-)}] = \frac{k\sqrt{-\Lambda}}{2\pi} \int_M 2\,\mathrm{Tr}\left\{ e \wedge (d\omega + \omega \wedge \omega) - \frac{\Lambda}{3} e \wedge e \wedge e - \frac{1}{2} d(\omega \wedge e) \right\} \,. \tag{2.70}$$

Then, setting $k = \frac{1}{4G\sqrt{-\Lambda}} = \frac{-i}{4G\sqrt{\Lambda}}$,

$$S[A^{(+)}, A^{(-)}] = I_{\text{EH}}[e, \omega] + (\text{boundary terms})\,. \tag{2.71}$$

This proves the classical equivalence between $S[A^{(+)}, A^{(-)}]$ and the Einstein-Hilbert action. If the last step seems unclear, check (2.66) and (2.68) and compare with (2.26).

Relationship between equations of motion. The equations of motion of the Chern-Simons formulation are

$$\left.\begin{array}{l} dA^{(+)} + A^{(+)} \wedge A^{(+)} = 0 \\ dA^{(-)} + A^{(-)} \wedge A^{(-)} = 0 \end{array}\right\}\,. \tag{2.72}$$

If we add them, we get

$$d\omega + \omega \wedge \omega - \Lambda e \wedge e = 0\,, \tag{2.73}$$

or, equivalently,

$$d\omega_a + \frac{1}{2}\varepsilon_{abc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\varepsilon_{abc}e^a \wedge e^b \tag{2.74}$$

Subtracting them, on the other hand, gives

$$de + \omega \wedge e + e \wedge \omega = 0\,, \tag{2.75}$$

which, using (2.67), is equivalent to

$$de_a + \varepsilon_{abc}\omega^b \wedge e^c = 0\,. \tag{2.76}$$

These are exactly the equations of motion that we got out of the Palatini action —cf. (2.30) and (2.29).

Relationship between symmetries. In the $(A^{(+)}, A^{(-)})$ picture, consider the infinitesimal gauge transformation generated by

$$u^{(\pm)}(x) = \rho_a^{(\pm)}(x) L^a\,. \tag{2.77}$$

The variation in the gauge fields would be

$$\begin{aligned} \delta A_\mu^{(\pm)} &= \mathcal{D}_\mu^{(\pm)} u^{(\pm)} = \partial_\mu u^{(\pm)} + [A^{(\pm)}, u^{(\pm)}] \\ &= \left(\partial_\mu \rho_a^{(\pm)}\right) L^a + \varepsilon_{abc}\left(\omega_\mu^b \pm \sqrt{-\Lambda} e_\mu^b\right)\rho^{(\pm)c} L^a\,. \end{aligned} \tag{2.78}$$

In terms of ω^a and e^a:

$$\begin{aligned} \delta\omega_\mu^a = \frac{1}{2}\left(\delta A_\mu^{(+)a} + \delta A_\mu^{(-)a}\right) &= \partial_\mu \frac{1}{2}(\rho^{(+)a} + \rho^{(-)a}) \\ + \varepsilon^{abc}\omega_{b\mu}\frac{1}{2}(\rho_c^{(+)} + \rho_c^{(-)}) &- \Lambda\,\varepsilon^{abc} e_{b\mu}\,\frac{1}{2\sqrt{-\Lambda}}\,(\rho_c^{(+)} - \rho_c^{(-)})\,, \end{aligned} \tag{2.79}$$

$$
\begin{aligned}
\delta e^a_\mu = \frac{1}{2\sqrt{-\Lambda}}\left(\delta A^{(+)a}_\mu - \delta A^{(-)a}_\mu\right) = \partial_\mu \frac{1}{2\sqrt{-\Lambda}}\left(\rho^{(+)a} - \rho^{(-)a}\right) \\
+ \varepsilon^{abc}\omega_{b\mu}\frac{1}{2\sqrt{-\Lambda}}\left(\rho^{(+)}_c - \rho^{(-)}_c\right) - \varepsilon^{abc}e_{b\mu}\,\frac{1}{2}(\rho^{(+)}_c - \rho^{(-)}_c)\,,
\end{aligned}
\tag{2.80}
$$

These transformations are of the form (2.32) + (2.33) with

$$
\rho^a = \frac{1}{2\sqrt{-\Lambda}}\left(\rho^{(+)}_c - \rho^{(-)}_c\right), \quad \tau^a = \frac{1}{2}(\rho^{(+)}_c + \rho^{(-)}_c)\,. \tag{2.81}
$$

Therefore, as in the $\Lambda = 0$ case, the gauge transformations of the Chern-Simons theory are equivalent, on-shell, to infinitesimal diffeomorphisms.

3 Chern-Simons theory and dS_3 quantum gravity

The first part of this essay is written from an entirely classical point of view, raising the question of whether the relationship between three-dimensional gravity and Chern-Simons theory extends to the quantum level. Non-pertubatively, the path integrals of the two theories seem to have very distinct features, but the equivalence could still hold at each order in perturbation theory like AdS/CFT does. The objective of the present section is to test this hypothesis by performing some explicit computations in dS_3 quantum gravity.

Section 3.1 introduces the reader to dS_3 quantum gravity, putting later computations into context and describing the most important features of dS_3 spacetime. The actual computations are performed over the course of sections 3.2 and 3.3, with the former discussing tree-level results and the latter extending the analysis to the one-loop level.

This section draws heavily on [4].

3.1 Introducing dS_3 quantum gravity

The term "de Sitter gravity" refers to any theory of gravity whose classical action is the Einstein-Hilbert action with positive cosmological constant. The name comes from the maximally symmetric solution of this action, de Sitter space, which finds widespread application as one of the simplest models of a universe in accelerated expansion. Despite its relevance for theoretical cosmology, our understanding of de Sitter quantum gravity is very limited; partly because it is hard to construct string theory solutions that contain a solution of Einstein's equations with positive cosmological constant, and partly due to the absence of a holographic description.

The situation is somewhat different in three spacetime dimensions. In this case, the Chern-Simons formulation discussed in the previous section can offer some hope for de Sitter (dS) quantum gravity. Recall that the 3d Einstein-Hilbert action is equivalent to a Chern-Simons theory based on the group $SO(1,3)$ when $\Lambda > 0$ and $SO(2,2)$ when $\Lambda < 0$. Most standard Chern-Simons methods require a compact gauge group, which is problematic because neither of these groups is compact. The advantage of the $\Lambda > 0$ case in this scenario is that a Wick rotation turns $SO(1,3)$ into the compact $SO(4)$. This feature makes the analysis significantly easier in dS_3, giving it the edge over its cousin AdS_3 —at least in a Chern-Simons context.

3.1.1 Contextualizing the computation

In order to define quantum field theory in a *fixed* curved background, dS_3 in this case, it is necessary to choose a vacuum state. The canonical choice is the *Hartle-Hawking (or Euclidean) vacuum state*, defined by analytic continuation from Euclidean signature. To make a long story short, field theory correlation functions are first computed on the sphere S^3 —the natural Euclidean continuation of Lorentzian dS_3— and then converted to Lorentzian signature via analytic continuation. This gives field theory expectation values in a particular vacuum state: the Hartle-Hawking vacuum state.

The Hartle-Hawking state is defined by a path integral in Euclidean signature. When gravity is neglected, the path integral is performed over a fixed background geometry —as in the previous paragraph. But if we allow for a dynamical spacetime, all solutions to the (Euclidean) gravitational equations of motion start contributing to the path integral, and therefore to the definition of the Hartle-Hawking state. The contribution of the different classical solutions is usually entered into the path integral as an infinite *sum over geometries*.

In this picture, the gravitational partition function is given by the norm of the Hartle-Hawking state, so we expect this function to also feature a sum over geometries. This leads to the following proposal for the partition function of dS quantum gravity in Euclidean signature:

$$Z = \sum_M \int [\mathcal{D}g]_M \, e^{-I_E[M,g]} \equiv \sum_M Z_M \,, \tag{3.1}$$

where

- $I_E[M,g]$ is the Euclidean dS gravity action (usually the Euclidean $\Lambda > 0$ Einstein-Hilbert action),
- the brackets indicate that we are integrating over equivalence classes of metrics under the action of the diffeomorphism group, and
- the sum is performed over the set $\{(M, g^{(M)})\}$ of solutions to the Euclidean equations of motion.

The geometries that we are summing over are, by definition, saddle points of I_E, so we can approximate

$$Z_M = \exp\left(-kI_E^{(0)} + I_E^{(1)} + \frac{1}{k}I_E^{(2)} + \dots\right)\bigg|_{(M,\,g^{(M)})} . \tag{3.2}$$

Here $I_E^{(i)}[M, g^{(M)}]$ denotes the quantum correction to the action at i-th order in perturbation theory. Observe that we have made the dependence of the dimensionless coupling constant $k = l/4G$ (the dS radius in Planck units) explicit by taking the powers of k out of the definition of the corrections.

For this approach to be successful, we should be able to:

1. Identify the (potentially infinite) set of classical solutions $\{(M, g^{(M)})\}$.
2. Compute the quantum corrections $I_E^{(i)}$ around each classical saddle, at all orders.

The plan for the next two sections is to carry out all of task 1 —in section 3.2.1— and part of task 2 —in the remaining sections. For task 2, we aim to evaluate the zeroth-order (*tree-level action*) and first-order (*one-loop determinant*) corrections on two classical saddles: $M = S^3$ and $M = L(p,q)$. In each case, the results obtained in the metric formulation will be compared to their Chern-Simons analogues. But first, let's review some basic features of dS_3 spacetime.

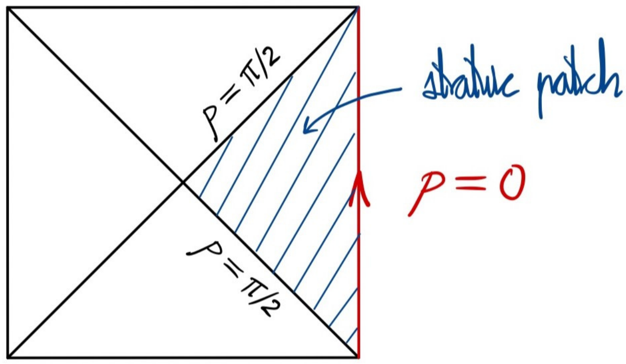

Figure 1. *A static patch on the Penrose diagram of dS_3. The observer that defines the patch travels along the $\rho = 0$ geodesic, measuring a causal horizon at $\rho = \pi/2$. Adapted from [5].*

3.1.2 Basic features of dS_3 spacetime

The material for this section is sourced from [5, §2.1].

Three-dimensional de Sitter space can be viewed as a timelike hyperbola embedded in 4-dimensional Minkowski space:

$$\eta_{AB} X^A X^B = l^2, \quad \eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)\,, \tag{3.3}$$

where $A, B \in \{0, 1, 2, 3\}$ and l is the *de Sitter radius*. From this definition, it is clear that the isometry group of dS_3 is $\mathrm{SO}(1, 3)$.

de Sitter space pictures an universe in accelerated expansion. In this universe, an observer moving along a timelike geodesic will see distant objects recede with a speed proportional to their distance to the observer. Given enough time (and ignoring bounding interactions like gravity), distant objects will lose causal contact with our observer, becoming hidden behind an horizon. The region of spacetime in causal contact with such an observer, depicted in Fig. 1, is called the *static patch*. A possible coordinate system for this region is

$$\left.\begin{aligned} X^0 &= l \cos\rho \sinh\tau \\ X^1 &= l \cos\rho \cosh\tau \\ X^2 &= l \sin\rho \cos\varphi \\ X^3 &= l \sin\rho \sin\varphi \end{aligned}\right\} \tag{3.4}$$

with $\tau \in (-\infty, \infty)$, $\rho \in [0, \pi/2)$ and $\varphi \in [0, 2\pi)$. This system has the observer sitting at $\rho = 0$, while their causal horizon lies at $\rho = \pi/2$. In these coordinates, the (induced) metric becomes:

$$ds^2 = \eta_{AB} dX^A dX^B = -l^2 \cos^2\rho\, d\tau^2 + l^2 d\rho^2 + l^2 \sin^2\rho\, d\varphi^2\,. \tag{3.5}$$

But remember, the Hartle-Hawking state is defined as a Euclidean path integral, so we need to switch to Euclidean signature. Euclidean de Sitter space can be defined through the global Wick rotation $X^0 \to X_E^0 = iX^0$, giving a 3-sphere $\delta_{AB} X_E^A X_E^B = l^2$, yet this seems overly restrictive. After all, we observers are only interested in correlation functions defined over our static patch, for that is everything that we may ever know about. Under this point of view, it suffices to perform the Wick rotation in the region corresponding to the static patch. A look back to (3.4) reveals that —thanks to the identity $\sinh(i\tau) = i\sin\tau$— the change $X^0 \to X_E^0 = iX^0$ can be achieved by means of $\tau \to \tau_E = i\tau$; exactly what the observer defining the patch would call a "Wick rotation". Under this transformation, the Lorentzian static patch metric becomes

$$\frac{ds^2}{l^2} = \cos^2\rho \, d\tau_E^2 + d\rho^2 + \sin^2\rho \, d\varphi^2 \,, \tag{3.6}$$

which has the same form as the round metric on S^3 in Hopf coordinates. Regularity at the horizon $\rho = \pi/2$ requires $\tau_E \sim \tau_E + 2\pi$, consistent with τ_E being an angular coordinate in S^3.

Lastly, if global Euclidean de Sitter is just the 3-sphere with the round metric, then its isometry group must be SO(4). A useful way to think about the isometries of the 3-sphere is through the isomorphism $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, where the two copies of SU(2) represent the left and right multiplication actions of $\mathrm{SU}(2) \cong S^3$ on itself and the $\mathbb{Z}^2$ quotient arises because $(\mathbf{1}_2, \mathbf{1}_2)$ and $(-\mathbf{1}_2, -\mathbf{1}_2)$ act in the same (trivial) way. At the level of Lie algebras, this isomorphism identifies $\mathfrak{so}(4)$ —the Wick-rotated $\mathfrak{so}(1,3)$— with $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

3.2 Tree-level partition function

The ultimate goal of this section is to show that the Chern-Simons formulation of Euclidean dS_3 gravity correctly reproduces the tree-level action of a graviton on S^3 and $L(p,q)$. This objective is achieved in three steps. First, in section 3.2.1, we identify the classical saddles of Euclidean dS_3. Next, in section 3.2.2, we use the metric formulation to set benchmark results. Finally, in section 3.2.3, we compare these results to those obtained using the Chern-Simons formulation.

3.2.1 Classical saddles

For a theory described by the Lorentzian action $I[g]$, the effect of a Wick rotation $t \to t_E = it$ on the path integral is expressed as

$$Z_M = \int [\mathcal{D}g]_M e^{iI[g]} = \int [\mathcal{D}g]_M e^{-I_E[g]} \,, \tag{3.7}$$

where the "Euclidean action" $I_E[g] := -iI[g]$ is another action of the metric that is defined integrating over the new Euclidean coordinates. In our case, $I[g]$ is the Einstein-Hilbert action with $\Lambda = 1/l^2 > 0$, which gives

$$I_E = -\frac{1}{16\pi G}\int_M d^3x_E\sqrt{g_E}\left(R - \frac{2}{l^2}\right) \equiv -\frac{1}{16\pi G}\int_M d^3x\sqrt{g}\left(R - \frac{2}{l^2}\right) \tag{3.8}$$

The change to Euclidean signature can be regarded as a coordinate change in a complex coordinate space, so the equations of motion —computed in section 2.1—, which are written in coordinate-invariant form, are still

$$R_{\mu\nu} = \frac{2}{l^2}g_{\mu\nu}\,. \tag{3.9}$$

This is equivalent to requiring that solutions are locally isometric to S^3. These geometries —collectively referred to as *spherical 3-manifolds*— have been fully classified in the literature [13]: they are of the form S^3/Γ, with Γ a discrete, freely acting subgroup of SO(4), the isometry group of S^3. There is an infinite and countable number of choices for Γ, thus confirming our prior guess that the set of classical solutions of Euclidean dS_3 gravity could be infinite. In the rest of this essay I will focus on two saddles in particular: the 3-sphere S^3 and the lens spaces $L(p,q)$.

The *lens spaces* can be defined as the family of spherical 3-manifolds S^3/Γ with cyclic Γ [8]. They are denoted by $L(p,q)$, where $p \in \mathbb{N}\cup\{0\}$ indicates the order of the group Γ and $q \in \{0,1,\ldots,p-1\}$ coprime with p represents different ways in which $\Gamma = \mathbb{Z}_p$ can act on S^3. More specifically, the lens space $L(p,q)$ can be constructed by considering the following identification on $S^3 \cong SU(2)$ [4]:

$$SU(2) \ni g \sim LgR\,, \tag{3.10}$$

with $(L,R) \in (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$ defined as

$$L = \begin{pmatrix} \omega_p^{(1+q)/2} & 0 \\ 0 & \omega_p^{-(1+q)/2}\end{pmatrix}, \quad R = \begin{pmatrix} \omega_p^{(1-q)/2} & 0 \\ 0 & \omega_p^{-(1-q)/2}\end{pmatrix}, \quad \omega_p = e^{2\pi i/p}\,. \tag{3.11}$$

This generates a $\mathbb{Z}_p$ quotient because (L,R) is a p-th root of unity in $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$. Although the lens spaces inherit the round metric from S^3, they will have in general different periodic identifications on the coordinates (τ_E,φ). While in the 3-sphere the identifications are

$$(\tau_E,\varphi) \sim (\tau_E,\varphi) + 2\pi(m,n) \quad \forall\; n,m\in\mathbb{Z}\,, \tag{3.12}$$

in $L(p,q)$ they are

$$(\tau_E,\varphi) \sim (\tau_E,\varphi) + 2\pi\left(\frac{m}{p},\, m\frac{q}{p}+n\right) \quad \forall\; n,m\in\mathbb{Z}\,. \tag{3.13}$$

[8] Another characterization of lens spaces is that they are the spherical 3-manifolds S^3/Γ with Γ abelian. This is because the discrete, freely acting subgroups of SO(4) are either cyclic or a central extension of a dihedral, tetrahedral, octahedral or icosahedral group [4].

Remark. Observe that the 3-sphere is technically a lens space: the trivial $L(1,0) = S^3$.

In the $G \ll l$ limit (so $k \gg 1$), the zeroth order term —the on-shell action— dominates the sum in (3.2). As we will see in the next section, the on-shell action evaluated on a saddle M is proportional to minus the volume of M. As a consequence, the 3-sphere will be the leading contribution to the sum over geometries, matching our intuition that S^3 is the "natural" Euclidean continuation of dS_3 and justifying that we elevate this saddle over the rest.

Remark. Not all saddles contribute to the Lorentzian path integral; only those that lie on the contour of stationary phase after rotating to Euclidean signature. This essay does not attempt to give a precise answer to the question of which saddles actually contribute. A possible way to motivate the inclusion of S^3 (resp. of the lens spaces $L(p,q)$) in the path integral would be to argue that correlation functions defined by analytic continuation from this saddle describe a canonical ensemble state at fixed temperature (resp. grand canonical ensemble state at fixed temperature and angular potential) —see [4, §2.1].

3.2.2 Metric formulation computation

In the metric formulation, the on-shell action of a generic saddle can be obtained by substituting (3.9) into the Euclidean action (3.8):

$$kI_E^{(0)}[M, g^{(M)}] \equiv I_E[M, g^{(M)}] = -\frac{1}{4\pi G}\int_M d^3x\sqrt{g} = -\frac{k}{\pi}\,\mathrm{Vol}(M)\,, \tag{3.14}$$

For $M = S^3/\Gamma$, since $|\Gamma| < \infty$ and Γ is acting freely on S^3:

$$I_E^{(0)}[M, g^{(M)}] = -\frac{k}{\pi}\frac{\mathrm{Vol}(\mathrm{S}^3)}{|\Gamma|} = -\frac{2\pi k}{|\Gamma|}\,. \tag{3.15}$$

The full zeroth-order partition function is, therefore,

$$Z^{(0)} = \sum_M e^{-I_E^{(0)}[M, g^{(M)}]} = \sum_{S^3/\Gamma} \exp\left(\frac{2\pi k}{|\Gamma|}\right)\,. \tag{3.16}$$

In particular, for $M = S^3 = S^3/1$ and $M = L(p,q) = S^3/\mathbb{Z}_p$:

$$Z_{S^3}^{(0)} = e^{2\pi k}\,, \quad Z_{L(p,q)}^{(0)} = e^{2\pi k/p}\,. \tag{3.17}$$

As we had anticipated, the contribution from the 3-sphere dominates the sum because the on-shell action is proportional to the volume of the saddle. The contributions from other saddles —including the lens spaces— include a factor of $1/|\Gamma|$ in the exponential and hence will be suppressed by factors which are exponentially large in k. Since $k \to \infty$ ($G \to 0$) is the semiclassical limit of the theory, we can interpret the latter as low energy quantum gravitational effects [4, p. 10].

3.2.3 Comparison with Chern-Simons

We conclude the tree-level analysis by reproducing the above results using Chern-Simons methods.

As noted in section 2, the whole Chern-Simons formulation of three-dimensional gravity relies in our ability to construct a 1-form ω^a from the spin connection ω_{ab}, for which we resort to the volume form ε_{abc}. In our conventions (outlined in the "Conventions and notation" section), the behaviour of the volume form is slightly dependent on the signature of the spacetime metric: while the volume form itself is always defined as $\varepsilon_{abc} = \sqrt{|g|}\epsilon_{abc}$, its "all indices raised" version ε^{abc} takes the form $\varepsilon^{abc} = \pm\frac{1}{\sqrt{|g|}}\epsilon_{abc}$, where the plus sign corresponds to the Riemannian case and the minus sign corresponds to the Lorentzian one. This sign difference reflects in the contraction identities for the volume form —see (0.6)— and ultimately propagates[9] into the defining equations of the first order formalism. In summary, in the Euclidean (Riemannian) case:

- the relationship between the dreibein and the metric is not (2.19), but

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}\,; \tag{3.18}$$

- the Palatini action is no longer (2.26), but

$$I_{\text{EH}}[e,\omega] = \frac{1}{8\pi G}\int_M \left\{ e^a \wedge \left(d\omega_a - \frac{1}{2}\varepsilon_a{}^{bc}\omega_b \wedge \omega_c \right) - \frac{\Lambda}{6}\varepsilon_{abc} e^a \wedge e^b \wedge e^c \right\}\,; \tag{3.19}$$

  and, as a consequence,

- the equations of motion (2.29)–(2.30) turn into

$$\left.\begin{array}{l} de_a - \varepsilon_{abc}\omega^b \wedge \omega^c = 0 \\ d\omega_a - \frac{1}{2}\varepsilon_{abc}\omega^b \wedge \omega^c = \frac{\Lambda}{2}\varepsilon_{abc} e^b \wedge e^c \end{array}\right\}\,. \tag{3.20}$$

Despite having previously discussed the Lorentzian case in section 2.4.2, the extra minus sign spotted above makes the direct formulation of Euclidean dS_3 gravity as a Chern-Simons theory subtle. A possible alternative approach, proposed by Witten [15], is to perform the Wick rotation on the Chern-Simons side instead of in the gravity side. This requires generalizing the Chern-Simons formulation of Lorentzian dS_3 gravity to allow for complex Chern-Simons levels.

Chern-Simons formulation of Lorentzian dS_3 gravity using complex levels [5]. Consider a cosmological constant $\Lambda = 1/l^2 > 0$. Choose a basis $\{L_a\}$ for $\mathfrak{su}(2)$. Define a pair

[9] At first sight, it may seem as if removing all extra minus in the Riemannian case has no net effect. The important thing to notice is that, if the dual spin connection is defined as $\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}$, the inverse relation now is $\omega_{ab} = \varepsilon_{abc}\omega^c$ and *not* $\omega_{ab} = -\varepsilon_{abc}\omega^c$.

of SU(2) gauge fields $A^{(\pm)} = A^{(\pm)a} L_a$ as

$$A^{(\pm)a} = \omega^a \pm \sqrt{-\Lambda}\, e^a = \omega^a \pm i\, \frac{e^a}{l} \,. \tag{3.21}$$

Let these fields have action

$$S[A^{(+)}, A^{(-)}] = S_{\mathrm{CS}}[A^{(+)}] + S_{\mathrm{CS}}[A^{(-)}] \,. \tag{3.22}$$

So far, we are following the same steps as in section 2.4.2. Now, as a novelty, suppose that the Chern-Simons levels of this theory can take complex values, $k_+,\, k_- \in \mathbb{C}$. Such debauchery comes at a price: not all values of the levels lead to a unitary, well-defined gauge theory. As argued in [15], a parametrization of the levels that ensures that we get a theory with the right properties is

$$k_+ = \delta + is, \quad k_- = \delta - is \quad \text{with} \quad \delta \in \mathbb{Z},\, s \in \mathbb{R} \,. \tag{3.23}$$

Plugging these levels into the action (3.22) and setting $s = -l/4G \equiv -k$ gives

$$S[A^{(+)}, A^{(-)}] = I_{\mathrm{EH}}[e,\omega] + \delta\, I_{\mathrm{GCS}}[e,\omega] + (\text{boundary terms}) \,, \tag{3.24}$$

where $I_{\mathrm{EH}}[e,\omega]$ denotes the *Lorentzian* Palatini action (2.26) and

$$I_{\mathrm{GCS}}[e,\omega] = \frac{1}{2\pi}\int_M \mathrm{Tr}\left\{\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right\} - \frac{1}{2\pi l^2}\int_M \mathrm{Tr}(e\wedge T) \tag{3.25}$$

is the so-called *gravitational CS action* (here T is the torsion 2-form defined in (2.29)). Observe that the addition of this term to the first-order gravitational action does not change the equations of motion. This settles the equivalence between the first order formalism of 3d general relativity and the Chern-Simons theory that we have just defined.

In fact, our current purposes do not need of the full generality of (3.23); we can set $\delta = 0$ and still proceed. In other words, it suffices to consider

$$k_+ = -ik, \quad k_- = ik \,. \tag{3.26}$$

Explicit form of $A^{(\pm)}$ for the lens space geometries. Equation (3.21) hence establishes a correspondence between solutions of Einstein's equations with positive cosmological constant and classical Chern-Simons solutions. Our next task is to use this correspondence to find the explicit form of $A^{(\pm)}$ for the lens space geometries. In this segment, $\eta_{ab} = \mathrm{diag}(+1,+1,-1)$.

Recall that the Lorentzian static patch metric is given by

$$\frac{ds^2}{l^2} = d\rho^2 + \sin^2\rho\, d\varphi^2 - \cos^2\rho\, d\tau^2 \,. \tag{3.27}$$

This metric can be written in terms of the dreibein

$$e^1 = l\,d\rho\,, \quad e^2 = l\sin\rho\,d\varphi\,, \quad e^3 = l\cos\rho\,d\tau\,. \tag{3.28}$$

A choice of dreibein determines a torsion-free spin connection through equation (2.29). For the dreibein defined above, (2.29) implies

$$\left.\begin{aligned} \omega^2\wedge e^3 &= \omega^3\wedge e^2 \\ \omega^1\wedge e^3 - \omega^3\wedge e^1 &= \tfrac{\cos\rho}{l\sin\rho}e^1\wedge e^2 \\ \omega^2\wedge e^1 - \omega^1\wedge e^2 &= \tfrac{\sin\rho}{l\cos\rho}e^1\wedge e^3 \end{aligned}\right\}\,, \tag{3.29}$$

which we can solve to get

$$\omega^1 = 0\,, \quad \omega^2 = -\sin\rho\,d\tau\,, \quad \omega^3 = \cos\rho\,d\varphi\,. \tag{3.30}$$

Substituting the dreibein and the spin connection into (3.21):

$$A^{(\pm)} = \pm iL_1 d\rho + (-\sin\rho\,d\tau \pm i\sin\rho\,d\varphi)L_2 + (\cos\rho\,d\varphi \pm i\cos\rho\,d\tau)L_3 \tag{3.31}$$

or, switching to Euclidean time $\tau_E = i\tau$,

$$A^{(\pm)} = \pm iL_1 d\rho + (\pm iL_2\sin\rho + L_3\cos\rho)d\theta_\pm\,, \tag{3.32}$$

where $\theta_\pm = \varphi \pm \tau_E$. It is straightforward to check that these connections are flat.

In [5], the shift to Euclidean signature is completed by a Wick rotation $L_3 \to L_3^E = iL_3$ of the algebra generators. If perform this operation on (3.32), we get

$$A^{(\pm)} = \pm iL_1 d\rho + i(\pm L_2\sin\rho - L_3^E\cos\rho)d\theta_\pm\,, \tag{3.33}$$

which matches their result.

Holonomies of $A^{(\pm)}$. The objects that we have just computed are flat connections on some principal bundle with structure group SU(2) and base manifold $L(p,q)$. As such, they are characterized by their holonomies around (homotopy classes of) non-contractible loops in $L(p,q)$ —see the discussion surrounding theorem 2.1.

More precisely, a flat SU(2) connection A on $L(p,q)$ can be characterized as an homomorphism $\rho_A\colon \pi_1(L(p,q)) \to \mathrm{SU}(2)$ that takes $[\gamma] \in \pi_1(L(p,q))$ to the holonomy $h_{A,\gamma}(r) \in \mathrm{SU}(2)$ of A around γ (where r is any element in the fibre of the basepoint of γ). The fundamental group of the lens space $L(p,q) = S^3/\mathbb{Z}_p$ is known to be $\pi_1(L(p,q)) = \mathbb{Z}_p$. Say then that this group is generated by a single non-contractible, order p loop that we denote by $[\gamma_0]$. Since the homomorphism ρ_A is fully determined by the holonomy of A around this loop, so is A. Moreover, the order of $\rho_A([\gamma_0])$ has to divide $|\mathbb{Z}_p| = p$, so $\rho_A([\gamma_0])$ must be (conjugate to) a rotation by an angle $2\pi n/p$ in SU(2) for some integer n which is defined modulo p. On the whole, specifying n suffices to fully determine the flat connection A.

Applied to $A^{(\pm)}$, the above argument implies that the round metric on $L(p,q)$ is characterized by the pair of mod p integers (n_+, n_-) which give the holonomies of (3.33). Let's calculate $n_\pm$.

In terms of the parameters $\theta_\pm = \varphi \pm \tau_E$ introduced in (3.33), the defining identifications of the lens space $L(p,q)$ —given in (3.13)— are

$$\theta_\pm \sim \theta_\pm + 2\pi \, \frac{m(q \pm 1) - np}{p} \qquad \forall \; n, \, m \in \mathbb{Z} \, . \tag{3.34}$$

The n identification is just the familiar $\varphi \sim \varphi + 2\pi$ identification of S^3, while the m identification induces the non-trivial $\mathbb{Z}_p$ quotient. The loop $[\gamma_0]$ can thus be taken to be the constant ρ cycle associated to the identification $(m,n) = (1,0)$. From (3.34), the integral of $d\theta_\pm$ around this cycle is

$$\oint_{\gamma_0} d\theta_\pm = 2\pi \frac{q \pm 1}{p} \, ; \tag{3.35}$$

hence

$$\oint_{\gamma_0} A^{(\pm)} = 2\pi \frac{q \pm 1}{p} i (\pm L_2 \sin\rho - L_3^E \cos\rho) \tag{3.36}$$

and we get

$$h_{\gamma_0, A^{(\pm)}} = \exp \oint A^{(\pm)} = \cos\left(\pi \frac{q \pm 1}{p}\right) + i(\pm\sigma_2 \sin\rho - \sigma_3 \cos\rho) \sin\left(\pi \frac{q \pm 1}{p}\right) \tag{3.37}$$

In writing this, I have assumed that $(L_1, L_2, L_3^E) = \frac{1}{2}(\sigma_1, \sigma_2, \sigma_3)$ [10] and used the formula

$$e^{i\alpha(\hat{\mathbf{n}}\cdot\sigma)} = I\cos\alpha + i(\hat{\mathbf{n}}\cdot\sigma)\sin\alpha \quad \forall \; \hat{\mathbf{n}} \in \mathbb{R}^3 \text{ with } |\hat{\mathbf{n}}| = 1 \, . \tag{3.38}$$

We see that, as expected, the holonomies in (3.37) are (conjugate to) rotations of angle $\frac{2\pi n_\pm}{p}$ with

$$(n_+, n_-) = \left(\frac{q+1}{2}, \frac{q-1}{2}\right) . \tag{3.39}$$

One should not worry too much about the fact that these are half-integer rather than integer; this happens because $A^{(\pm)}$ are not "proper" SU(2) connections themselves, but the result of decomposing a SO(4) connection through the splitting $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}^2$ [4].

On-shell action. Knowing the holonomies of $A^{(\pm)}$ enables us to compute the on-shell action of $L(p,q)$ in the Chern-Simons formulation, as we will see next.

The tree-level partition function of dS_3 quantum gravity on $L(p,q)$ can be written in terms of on-shell Chern-Simons actions as

$$Z^{(0)}_{L(p,q)} = \exp\left(-I^{(0)}_{\text{EH, Eucl.}}\right) \equiv \exp\left(iS^{(0)}\right) = \exp\left(iS^{(0)}_{\text{CS}}[A^{(+)}] + iS^{(0)}_{\text{CS}}[A^{(-)}]\right) , \tag{3.40}$$

[10] A factor of $-i$ is probably missing here.

where we have used that $I_{\text{EH, Eucl.}} = -iI_{\text{EH}}$ and the equivalence (3.24) (with $\delta = 0$).

The Chern-Simons invariant of a SU(2) gauge field A on a lens space $L(p, q)$ has been computed in the literature [10]. It is given as a function of the holonomy n of the field by

$$\frac{1}{8\pi^2}\int \mathrm{Tr}\left\{A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right\} = \frac{q^*}{p}n^2\,, \tag{3.41}$$

where $q^* \in \{1, \ldots, p-1\}$ is such that $q^*q = 1 \pmod p$. If we substitute this result into (3.40), we find

$$\begin{aligned} Z^{(0)}_{L(p,q)} &= \exp\left[\frac{2\pi i}{p}q^*(k_+n_+^2 - k_-n_-^2)\right] = \exp\left\{\frac{2\pi k}{p}q^*\left[\left(\frac{q+1}{2}\right)^2 - \left(\frac{q-1}{2}\right)^2\right]\right\} \\ &= \exp\left(\frac{2\pi k}{p}qq^*\right) = \exp\left(\frac{2\pi k}{p}\right)\,, \end{aligned} \tag{3.42}$$

which matches the benchmark result of section 3.2.2.

3.3 One-loop corrections

I would like to conclude with a brief report on the one-loop results from [4, §4]. I have split the discussion in three parts: definition of the one-loop partition function as a ratio of functional determinants —sec. 3.3.1—, computation of the determinants —sec. 3.3.2—, and comparison with Chern-Simons formulation —sec. 3.3.3. The reader should not expect this section to maintain the same level of detail as the rest of the essay; the aim here is rather to provide a broad overview of the whole calculation. The computational load is further alleviated by restricting some aspects of the analysis to the classical saddles $M = S^3$ and $M = L(p, 1)$.

3.3.1 One-loop partition function of Euclidean quantum gravity

Following [6], consider Euclidean gravity on a closed (compact and without boundary) spacetime of dimension three, M. The quantum theory is defined through the following functional integral of the spacetime metric:

$$\int [\mathcal{D}g]\, e^{-iI_E[g]}\,. \tag{3.43}$$

Suppose that the only relevant contributions to this functional integral can be expressed as small perturbations from a classical solution g^{cl}:

$$g_{\mu\nu}(x) = g^{\text{cl}}_{\mu\nu}(x) + h_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll |g^{\text{cl}}_{\mu\nu}|\,. \tag{3.44}$$

We can then expand the action in a functional Taylor series about $g^{\rm cl}$:

$$\begin{aligned} I_E[g^{\rm cl}+h] &= I_E[g^{\rm cl}] + \int_M d^3x \, \frac{\delta I_E}{\delta g_{\mu\nu}(x)}\bigg|_{g^{\rm cl}} h_{\mu\nu}(x) \quad (\to 0) \\ &+ \frac{1}{2!}\int_M d^3x d^3y \, \frac{\delta^2 I_E}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)}\bigg|_{g^{\rm cl}} h_{\mu\nu}(x) h_{\rho\sigma}(y) + \dots \\ &\equiv I_E[g^{\rm cl}] + \partial^2 I_E\big|_{g^{\rm cl}} h^2 + \dots . \end{aligned} \tag{3.45}$$

In this picture, the diffeomorphism invariance of the Einstein-Hilbert action translates into the (infinitesimal) gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}, \quad \xi \in \mathfrak{X}(M)\,, \tag{3.46}$$

where $\mathfrak{X}(M)$ is the space of all vector fields on M. Field configurations related by a gauge transformation describe equivalent physical settings, so in principle our functional integrals should be defined over the space of orbits of metric perturbations under the action of $\mathfrak{X}(M)$ ($\cong$ infinitesimal diffeomorphisms). A celebrated alternative to deal with gauge symmetry in functional integrals is the *Faddeev-Popov procedure.* In this procedure, gauge symmetry is broken by adding a gauge-fixing term $I_{\rm gauge}$ to the action, which takes the form of a functional determinant. Now, this functional determinant can usually be rewritten in terms of an action of some fictitious, fermionic fields, so the addition of $I_{\rm gauge}$ can be compensated by also including the action $I_{\rm ghost}$ of these "ghost" fields. In the present case, the ghost fields take the form of a spin-1 field V_μ and its complex conjugate V_μ^*. All things considered, the functional integral takes the form

$$\begin{aligned} \int [\mathcal{D}h]\, e^{-I_E[g_{\rm cl}+h]} = N \int \mathcal{D}h\, \mathcal{D}V\, \mathcal{D}V^* \exp\{-(I_E[g_{\rm cl}+h] + I_{\rm gauge}[g_{\rm cl}, h] \\ + I_{\rm ghost}[g_{\rm cl}, V, V^*])\}\ , \end{aligned} \tag{3.47}$$

where N is just a normalization factor.

One-loop Feynman diagrams correspond to terms quadratic (two vertices) in $h_{\mu\nu}$ and V_μ, so the one-loop action will consist of what we had schematically denoted $\partial^2 I_E\big|_{g^{\rm cl}} h^2$ as well as the quadratic part of $I_{\rm gauge}[g,h] + I_{\rm ghost}[g,V,V^*]$. A explicit computation of these contributions, which I will not perform here, would require decomposing the fields appearing in the Lagrangian into

$$\phi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{3} g_{\mu\nu} h^\alpha{}_\alpha\,, \quad \phi_\mu \equiv V_\mu\,, \quad \phi \equiv h^\alpha{}_\alpha\,. \tag{3.48}$$

Next, one would define the second-order differential operators (*Lichnerowicz Laplacians*)

$$\begin{aligned}
\Delta_{(2)}^{LL} T_{\mu\nu} &= -\Delta_{(2)} T_{\mu\nu} - 2R_{\mu\rho\nu\sigma} T^{\rho\sigma} + R_{\mu\rho} T_{\nu}^{\rho} + R_{\nu\rho} T_{\nu\rho} T_{\mu}^{\rho} \\
\Delta_{(1)}^{LL} T_{\mu} &= -\Delta_{(1)} T_{\mu} + R_{\mu\nu} T^{\nu} \,, \\
\Delta_{(0)}^{LL} T &= -\Delta_{(0)} T \,,
\end{aligned} \tag{3.49}$$

where $\Delta_{(j)} = \nabla^{\rho}\nabla_{\rho}$ is the usual Laplacian acting on fields of spin j. The one-loop Lagrangian would then admit an expression in terms of these operators and the fields of (3.48). After simplifying using the equations of motion $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$, the one-loop partition function would take the form of a product of Gaussian integrals over the fields $\phi_{\mu\nu}$, ϕ_{μ} and ϕ, which can be readily computed to yield [4]

$$Z^{(1)} = e^{-I_E^{(1)}} = \frac{\det\left(\Delta_{(1)}^{LL} - \frac{2}{3}R\right)}{\left[\det\left(\Delta_{(2)}^{LL} - \frac{2}{3}R\right)\det\left(\Delta_{(0)}^{LL} - \frac{2}{3}R\right)\right]^{1/2}} \,. \tag{3.50}$$

As pointed out in [4], if all the differential operators appearing in (3.50) have a positive definite spectrum, this expression simplifies to

$$Z^{(1)} = \sqrt{\frac{\det'\left(-\Delta_{(1)} - 2\right)_{T}}{\det'\left(-\Delta_{(2)} + 2\right)_{TT}}} \,, \tag{3.51}$$

Here the prime symbol means that determinants are being computed only with respect to positive eigenvalue subspaces. As for the subscript T (TT), it indicates that we are considering the transverse (transverse traceless) part of the corresponding operator.

Unfortunately, the assumption of positive-definiteness fails on the spherical 3-manifolds that we have been studying in this essay: the "scalar" and "vector" operators

$$\Delta_{(0)}^{LL} - \frac{2}{3}R \quad \text{and} \quad \Delta_{(1)}^{LL} - \frac{2}{3}R \tag{3.52}$$

will have non-positive eigenvalues. Let's study each of these separately.

The vector operator has no negative modes, but any Killing vector field will be a zero mode. Indeed, derivatives of a Killing vector (KV) K^{ρ} can be related to the Riemann tensor by [3]

$$\nabla_{\mu}\nabla_{\sigma}K^{\rho} = R^{\rho}{}_{\sigma\mu\nu}K^{\nu} \,. \tag{3.53}$$

Contracting this expression gives

$$\nabla_{\mu}\nabla_{\sigma}K^{\mu} = R_{\sigma\nu}K^{\nu} \,. \tag{3.54}$$

Because K^{μ} satisfies Killing's equation $\nabla_{(\mu}K_{\nu)} = 0$, this is the same as

$$\Delta_{(1)}^{LL} K_{\mu} = -\nabla^{\rho}\nabla_{\rho}K_{\mu} + R_{\mu\nu}K^{\nu} = \nabla^{\rho}\nabla_{\mu}K_{\rho} + R_{\mu\nu}K^{\nu} = 2R_{\mu\nu}K^{\nu} \,. \tag{3.55}$$

So, using that in three dimensions $R_{\mu\nu} = 2\Lambda g_{\mu\nu} = \frac{R}{3} g_{\mu\nu}$, we get

$$\Delta^{LL}_{(1)} K_\mu = \frac{2}{3} R\, K_\mu \,, \tag{3.56}$$

as we wanted to prove.

The problem with these zero modes is actually deeper than just not being able to carry out the simplification (3.51). In the Faddeev-Popov procedure, the gauge-fixing determinant takes the form

$$I_{\text{gauge}}[h] = \int_{\mathfrak{X}(M)} \mathcal{D}\xi \, \delta(\hat{h}^\xi - h) \,, \tag{3.57}$$

where $\hat{h}$ is some fixed reference metric perturbation and $\hat{h}^\xi_{\mu\nu} = \hat{h}_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}$ (cf. 3.46). This supposes that the functional $\xi \mapsto \hat{h}^\xi$ intersects each orbit of the diffeomorphism group exactly once. But this is clearly not true for ξ^μ equal to any KV K^μ; in this case we will have $\hat{h}^\xi = \hat{h}$ regardless of our choice of reference metric. And KVs generate isometries of the metric, which can be non-trivial elements of the diffeomorphism group, hence ignoring them is not an option. Luckily, this problem has an easy solution: all the spacetime manifolds under consideration have a compact isometry group, so we can simply replace

$$I_{\text{gauge}} \longrightarrow \frac{I_{\text{gauge}}}{k V_{\text{KV}}} \,, \tag{3.58}$$

where V_{KV} denotes the volume of the isometry group and the factor of k has been included to account for the normalization of the metric fluctuations.

Let's now shift our attention to the scalar operator in (3.52). On a spherical 3-manifold, the spectrum of this operator has at least one negative eigenvalue coming from the constant mode. In addition, a slight generalization of the argument used to prove that KVs are zero modes of the vector operator can be used to show that any conformal Killing vector field[11] (CKV) K^μ gives rise to a negative mode $T = \nabla_\alpha K^\alpha$. Negative modes obstruct the convergence of the path integral because they lead to Gaussian integrals with the wrong sign in the exponential. Happily for us, lens spaces have no conformal Killing vectors, so we do not need to worry about this issue.

We may then conclude that the one-loop partition function ("determinant") of a graviton on $L(p,q)$ is

$$Z^{(1)} = D_{\text{zm}} \sqrt{\frac{\det'\left(-\Delta_{(1)} - 2\right)}{\det'\left(-\Delta_{(2)} + 2\right)}} \,, \qquad D_{\text{zm}} = \frac{1}{k V_{\text{KV}}} \,. \tag{3.59}$$

3.3.2 Computation of the one-loop determinants

Our next task is to compute the one-loop determinants appearing in (3.59). We will use zeta function regularization and heat kernel techniques for this.

[11] Recall that a CKV is a vector field K^μ that satisfies the conformal Killing equation, which in dimension D takes the form

$$\nabla_{(\mu} K_{\nu)} - \frac{2}{D} g_{\mu\nu} \nabla_\rho K^\rho = 0 \,.$$

The functional determinant. The first question that we need to address is the definition of the functional determinant. Consider a linear operator $\bar{\Delta}$ with a positive definite spectrum $\{\lambda_n\}$ and corresponding degeneracies $\{d_n\}$. If $\bar{\Delta}$ acts on a finite-dimensional vector space, the determinant of $\bar{\Delta}$ is

$$\det(\bar{\Delta}) = \prod_n \lambda_n^{d_n} \tag{3.60}$$

or

$$\log\det(\bar{\Delta}) = \sum_n d_n \log(\lambda_n)\,. \tag{3.61}$$

The operators appearing in (3.59) are defined over infinite-dimensional spaces. Equation (3.61) might still serve as a heuristic definition of determinant in this case, but the right hand side is now an infinite series that will, in general, diverge. To regulate this sum, we define the zeta function $\zeta_{\bar{\Delta}}\colon \mathbb{C} \to \mathbb{C}$ as

$$\zeta_{\bar{\Delta}}(s) = \sum_n \frac{d_n}{\lambda_n^s} \tag{3.62}$$

in the domain of convergence of the sum —typically for positive and large enough s—, and as the analytic continuation of the sum elsewhere. In finite dimension, the domain of convergence of the sum is the entire complex plane, so we have

$$\frac{d}{ds}\zeta_{\bar{\Delta}}(0) = -\sum_n d_n \log(\lambda_n) = -\log\det(\bar{\Delta})\,. \tag{3.63}$$

Thus, a sensible *definition* for the functional determinant of a linear operator $\bar{\Delta}$ is

$$\log\det(\bar{\Delta}) \coloneqq -\frac{d}{ds}\zeta_{\bar{\Delta}}(0)\,. \tag{3.64}$$

Heat kernels. The problem of computing the one-loop determinants hence reduces to finding a suitable zeta function for each operator in (3.59). This is where heat kernels come in handy. Suppose that positive definite part of the spectrum of $\bar{\Delta}_{(j)} \equiv -\Delta_{(j)}$ is given by

$$\bar{\Delta}_{(j)}\psi_n^{(j)}(x) = \lambda_n^{(j)}\psi_n^{(j)}(x)\,, \tag{3.65}$$

where $\lambda_n^{(j)} > 0$ and $\{\psi_n^{(j)}\}$ form an orthonormal basis. The *heat kernel* of $\Delta_{(j)}$ is

$$K^{(j)}(x,y;t) \coloneqq \langle y|e^{t\Delta_{(j)}}|x\rangle = \sum_n \psi_n^{(j)}(x)\,\psi_n^{(j)}(y)^*\,e^{-\lambda_n^{(j)}t}\,. \tag{3.66}$$

As its name suggests, the heat kernel obeys the heat equation:

$$(\partial_t - \Delta^x_{(j)})K^{(j)}(x,y;t) = 0\,. \tag{3.67}$$

Integrating over space and using the orthonormality of $\{\psi_n^{(j)}\}$, we get

$$K^{(j)}(t) \equiv \int d^3x\sqrt{g}\, K^{(j)}(x,x;t) = \sum_n d_n^{(j)} e^{-\lambda_n^{(j)} t}\,, \tag{3.68}$$

where $d_n^{(j)}$ denotes the degeneracy of the eigenvalue $\lambda_n^{(j)}$. Observe that this function encodes all the information about the spectrum of $\bar{\Delta}_{(j)}$ that we need to define $\zeta_{\bar{\Delta}_{(j)}}$. To write $\zeta_{\bar{\Delta}_{(j)}} \equiv \zeta_{(j)}$ in terms of $K^{(j)}$, first notice that, for sufficiently large $s > 0$,

$$\begin{aligned}&\int_0^\infty t^{s-1} K^{(j)}(t)dt = \sum_n d_n^{(j)} \int_0^\infty t^{s-1} e^{-\lambda_n^{(j)} t} dt = \sum_n d_n^{(j)} \frac{\Gamma(s)}{(\lambda_n^{(j)})^s} = \zeta_{(j)}(s)\,\Gamma(s)\\ &\implies \zeta_{(j)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K^{(j)}(t)\,.\end{aligned} \tag{3.69}$$

By uniqueness of analytic continuation,

$$\zeta_{(j)}(s) \equiv \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K^{(j)}(t) \tag{3.70}$$

on all of $\mathbb{C}$.

We can use this zeta function to get a simple expression for the one-loop determinants of (3.59) in terms of the (integrated) heat kernels $K^{(j)}(t)$. Applying the zeta-regulated definition of functional determinant (3.64):

$$\log\left[\det{}'(-\Delta_{(j)} + m_j^2)\right] = -\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} K^{(j)}(t) e^{-m_j^2 t}\right)\Bigg|_{s=0}\,. \tag{3.71}$$

Hence we arrive at

$$\begin{aligned}\log\frac{Z^{(1)}}{D_{zm}} &= -\frac{1}{2}\log\left[\det{}'(-\Delta_{(2)}+2)\right] + \frac{1}{2}\log\left[\det{}'(-\Delta_{(1)}-2)\right]\\ &= \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K^{(2)}(t)\,e^{-2t}dt - \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K^{(1)}(t)\,e^{2t}\right)\Bigg|_{s=0}\,.\end{aligned} \tag{3.72}$$

This is great news, because the heat kernels $K^{(j)}(t)$ for the lens spaces have been explicitly computed by [8]. For example, for the 3-sphere:

$$K^{(j)}(t) = (2-\delta_{j,0}) \sum_{n=j+1}^{\infty} (n^2 - j^2) e^{-E_n^{(j)} t}\,, \tag{3.73}$$

where

$$E_n^{(j)} = n^2 - j - 1\,. \tag{3.74}$$

Recall that these kernels are of the form (3.68), so we may directly read $\lambda_n^{(j)}$ and $d_n^{(j)}$ from the expression above. The unregulated one-loop partition function for the sphere then is

$$\log \frac{Z^{(1)}_{S^3}}{D_{zm}} = -\sum_{n=3}^{\infty} \left[(n^2-4)\log(n^2-1) - (n^2-1)\log(n^2-4) \right] . \tag{3.75}$$

According to (3.62), the corresponding zeta function is constructed as the analytic continuation of

$$\zeta_{S^3}(s) = -\frac{1}{2}\zeta_{(2)}(s) + \frac{1}{2}\zeta_{(1)}(s) = -\sum_{n=3}^{\infty} \frac{n^2-4}{(n^2-1)^s} + \sum_{n=3}^{\infty} \frac{n^2-1}{(n^2-4)^s} , \tag{3.76}$$

but, noticing that $\log(n^2-1) = \log(n+1)+\log(n-1)$ and $\log(n^2-4) = \log(n-2)+\log(n+2)$ in (3.75), we can replace this sum by the more elegant

$$\zeta_{S^3}(s) = -\sum_{n=3}^{\infty} \left(\frac{n^2-4}{(n+1)^s} + \frac{n^2-4}{(n-1)^s} \right) + \sum_{n=3}^{\infty} \left(\frac{n^2-1}{(n+2)^s} + \frac{n^2-1}{(n-2)^s} \right) . \tag{3.77}$$

Dropping terms independent of s, the analytic continuation of this new sum is

$$\zeta_{S^3}(s) = 12\,\zeta(s) - \frac{2}{2^s} - \frac{3}{4^s} , \tag{3.78}$$

where ζ is the Riemann zeta function. The derivative of the Riemann zeta function has a well-known value at $s = 0$:

$$\frac{d}{ds}\zeta(0) = -\frac{1}{2}\log(2\pi) . \tag{3.79}$$

With this[12],

$$\log \frac{Z^{(1)}_{S^3}}{D_{zm}} = -\frac{d}{ds}\zeta_{S^3}(0) = \log \frac{\pi^6}{4} \implies Z^{(1)}_{S^3} = D_{zm}\frac{\pi^6}{4} . \tag{3.80}$$

Similar considerations would lead us to the zeta function and regulated one-loop determinants for the lens spaces $L(p,q)$. As an example, let me just cite here the results of [4] for the case $q = 1 \pmod p$. For $p > 2$, the zeta function takes the form

$$\zeta_{(p,1)}(s) = 2p^{-s}\left[\zeta\left(s, \frac{2}{p}\right) + \zeta\left(s, -\frac{2}{p}\right) - \left(-\frac{p}{2}\right)^s \right] + 8p^{-s}\,\zeta(s) - \frac{2}{4^s} - \frac{1}{2^s} , \tag{3.81}$$

where $\zeta(s,a)$ is the Hurwitz zeta function. This gives a regulated one-loop contribution of

$$Z^{(1)}_{(p,1)} = D_{zm}\frac{2\pi^4}{p^4}\sin^2\left(\frac{2\pi}{p}\right) , \quad p > 2 . \tag{3.82}$$

The special case $p = 2$ has to be treated separately. In this case, the zeta function is given by

$$\zeta_{(2,1)}(s) = \frac{12}{2^s}\,\zeta(s) - \frac{3}{4^s} - \frac{2}{2^s} , \tag{3.83}$$

[12] I was able to reproduce (3.78) by myself, but not (3.80). I get the wrong powers of 2.

and the partition function is

$$Z^{(1)}_{(2,1)} = D_{zm}\frac{\pi^6}{2^8}\,. \tag{3.84}$$

Zero modes. The last step of the computation is figuring out the prefactor D_{zm} that we had introduced to account for zero modes of the vector operator in (3.50). Recall that these modes were essentially the Killing vectors of the geometry under consideration, and that they had a net effect of multiplication by

$$D_{zm} = \frac{1}{kV_{\text{KV}}}\,. \tag{3.85}$$

Here V_{KV} is the volume of the isometry group, of which the Killing vectors are infinitesimal generators. For a spherical 3-manifold S^3/Γ,

$$V_{\text{KV}} \propto \left(\text{Vol}(S^3/\Gamma)\right)^{n_K/2} = \left(\frac{2\pi^2}{|\Gamma|}\right)^{n_K/2}\,, \tag{3.86}$$

where n_K is the number of Killing vectors of the geometry[13]. The 3-sphere, for example, has $n_K = 6$, as can be deduced from (3.73).

After multiplication by D_{zm}, the one-loop partition functions computed in (3.80), (3.82) and (3.84) become [4]:

$$Z^{(1)}_{S^3} = \frac{\pi^3}{2^5 k}\,; \tag{3.87}$$

$$Z^{(1)}_{(p,1)} = \frac{\pi}{2kp^2}\sin^2\left(\frac{2\pi}{p}\right)\,, \quad p > 2\,; \tag{3.88}$$

$$Z^{(1)}_{(2,1)} = \frac{\pi^3}{2^{11}k}\,. \tag{3.89}$$

For completeness, the one-loop determinant for a lens space $L(p,q)$ with $q \neq 1 \pmod p$ can also be computed by this method, and is [4]

$$Z^{(1)}_{(p,q)} = \frac{2\pi}{kp}\left[\cos\left(\frac{2\pi}{p}\right) - \cos\left(\frac{2\pi q}{p}\right)\right]\left[\cos\left(\frac{2\pi}{p}\right) - \cos\left(\frac{2\pi q^*}{p}\right)\right]\,. \tag{3.90}$$

Remark. In the above discussion we have only included isometries connected to the identity. Discrete isometries not connected to the identity contribute with (at most) an additional factor of four [4].

3.3.3 Comparison with Chern-Simons

In section 3.2.3 we argued that the Chern-Simons/gravity correspondence takes a lens space spacetime $L(p,q)$ to a pair of flat SU(2) connections over $L(p,q)$. A flat SU(2) connection over $L(p,q)$ is characterized by a (half-)integer n that gives the holonomy of the connection

[13] The authors of [4] claim equality in this expression; however, their later results do not correspond to just dividing by k times this factor.

around the non-contractible cycle of the lens space. In the large k limit, the contribution to the SU(2) Chern-Simons partition function of one such connection is [10, 4]

$$Z_{\rm CS} \approx i\sqrt{\frac{2}{k_{\rm CS}\,p}}\sum_{n=1}^{p}\exp\left(2\pi i k_{\pm}\frac{q^{*}n^{2}}{p}\right)\sin\left(2\pi\frac{q^{*}n}{p}\right)\sin\left(2\pi\frac{n}{p}\right)\,. \tag{3.91}$$

The joint large-k partition function of the two connections can be constructed by applying the expression above to the Chern-Simons action (3.22). The contribution of the connections $(A^{(+)}, A^{(-)})$ with holonomies (3.37) to this partition function is then seen to be:

$$Z_{(p,q)} = \frac{1}{2kp}e^{2\pi k/p}\left[\cos\left(\frac{2\pi}{p}\right) - \cos\left(\frac{2\pi q}{p}\right)\right]\left[\cos\left(\frac{2\pi}{p}\right) - \cos\left(\frac{2\pi q^{*}}{p}\right)\right] \tag{3.92}$$

Taking the large k limit kills all corrections at order two-loops or higher, so the result can be interpreted as the tree and one-loop level contributions of the $L(p,q)$ connections to the action, i.e.,

$$Z_{(p,q)} = \exp(-kS^{(0)} + S^{(1)})\,. \tag{3.93}$$

Up to numerical factors that are independent of p and q, equation (3.92) agrees with the metric formulation results —cf. (3.17) and (3.90). This remarkable convergence of results confirms that, at least up to one-loop level, the Chern-Simons formulation of 3d general relativity can be leveraged to a legitimate definition of perturbative quantum gravity.

4 Conclusion

One of the barriers that we find in our quest for a theory of quantum gravity is that we only have a partial understanding of the role of spacetime topology. Due to the emphasis that standard general relativity places on the metric degrees of freedom, this issue may have not received the attention that it deserves. This essay has tried to shed some light on this question by studying the Chern-Simons formulation of three-dimensional gravity. In a nutshell, this formulation uses a purely topological description of Einstein's theory to provide a perturbative definition of quantum gravity.

As I tried to convey at the beginning of section 2, the role of spacetime topology is specially clear in dimension three due to the absence of propagating degrees of freedom. Indeed, in three dimensions, the first-order Einstein-Hilbert action happens to be equivalent to the action functional of a Chern-Simons theory, which is a topological theory. The first part of the essay was devoted to developing and studying this classical-level correspondence. A key observation was that the gauge group of the Chern-Simons theory could be taken to be the isometry group of the maximally symmetric solution of the gravity action. In particular, the gauge group depended on whether we considered a negative, zero or positive cosmological constant in the Einstein-Hilbert action. For zero cosmological constant, the gauge group was simply the Poincaré group $\mathrm{ISO}(2,1)$. For cosmological constant $\Lambda \neq 0$, a clearer picture was obtained by splitting the algebra of the isometry group into $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, for $\Lambda < 0$, or $(\mathfrak{su}(2) \times \mathfrak{su}(2))_{\mathbb{C}}$, for $\Lambda > 0$ —note the further complexification in the $\Lambda > 0$ case. In either case, our analysis drew a connection between the equations of motion and symmetries of the two theories. As one would expect, the gauge transformations of the Chern-Simons theory are related to infinitesimal diffeomorphisms on the gravity side.

Chern-Simons theories have a relatively straightforward quantization. It is therefore natural to wonder if the Chern-Simons formulation of three-dimensional gravity can be used to give a definition of quantum gravity, at least at the perturbative level. The objective of the second part of the essay —section 3— was to test this hypothesis in the context of dS_3 quantum gravity. We started by expressing the Euclidean path integral of dS_3 gravity as a sum of functional Taylor series about different classical solutions of the equations of motion ("sum over geometries"). We were able to identify all possible solutions: they are of the form S^3/Γ with Γ a discrete, freely acting subgroup of $\mathrm{SO}(4)$, and go under the name of spherical 3-manifolds. Not all of these solutions contribute to the path integral, so we decided to focus on a subset whose inclusion can be motivated on physical grounds: the lens spaces $L(p,q) = S^3/\mathbb{Z}^p$. By a direct gravity computation, the tree-level contribution of these geometries to the partition function was shown to be

$$Z^{(0)}_{L(p,q)} = e^{2\pi k/p} \,. \tag{4.1}$$

This result was then reproduced using Chern-Simons methods. The identification of the classical saddles and computation of tree-level contributions was done in section 3.2. For the next and final section, we extended the analysis to the one-loop level, although restricting part of the discussion to just S^3 and $L(p,1)$. Here we began by applying the Faddeev-Popov procedure to cast the one-loop partition function as a ratio of functional determinants. These determinants were then computed and regulated using heat kernel techniques. We finished by comparing our results to Chern-Simons analogues in the literature, finding total agreement between the two formulations. We may therefore conclude that the Chern-Simons formulation of three-dimensional gravity gives a definition of quantum gravity that is valid at least up to the one-loop level in perturbation theory.

References

[1] A. Achúcarro and P.K. Townsend. "A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories". In: *Physics Letters B* 180.1 (1986), pp. 89–92. DOI: 10.1016/0370-2693(86)90140-1.

[2] Steven Carlip. *Quantum Gravity in 2+1 Dimensions*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.

[3] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Addison Wesley, 2004.

[4] Alejandra Castro, Nima Lashkari, and Alexander Maloney. "A de Sitter Farey Tail". In: *Physical Review D* 83.12 (2011). DOI: 10.1103/physrevd.83.124027.

[5] Alejandra Castro et al. "Keeping matter in the loop in dS3 quantum gravity". In: *Journal of High Energy Physics* 2023.7 (2023). DOI: 10.1007/jhep07(2023)120.

[6] S.M. Christensen and M.J. Duff. "Quantizing gravity with a cosmological constant". In: *Nuclear Physics B* 170.3 (1980), pp. 480–506. DOI: 10.1016/0550-3213(80)90423-X.

[7] Geoffrey Compère and Adrien Fiorucci. *Advanced Lectures on General Relativity*. 2019. arXiv: 1801.07064 [hep-th].

[8] Justin R. David, Matthias R. Gaberdiel, and Rajesh Gopakumar. "The Heat Kernel on AdS3 and its Applications". In: *Journal of High Energy Physics* 2010.4 (2010). DOI: 10.1007/jhep04(2010)125.

[9] Paul Garrett. *Sporadic isogenies to orthogonal groups*. May 2015. URL: https://www-users.cse.umn.edu/~garrett/m/v/sporadic_isogenies.pdf.

[10] Lisa C. Jeffrey. "Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation". In: *Communications in Mathematical Physics* 147.3 (1992), pp. 563–604.

[11] B. Khesin and R. Wendt. *The Geometry of Infinite-Dimensional Groups*. A series of modern surveys in mathematics. Springer Berlin, Heidelberg, 2008. ISBN: 9783540772637. DOI: 10.1007/978-3-540-77263-7.

[12] C.H. Taubes. *Differential Geometry: Bundles, Connections, Metrics and Curvature*. Oxford Graduate Texts in Mathematics. OUP Oxford, 2011. ISBN: 9780191621222.

[13] William P. Thurston. *The Geometry and Topology of Three-Manifolds*. Mar. 2002. URL: https://www.math.unl.edu/~jkettinger2/thurston.pdf.

[14] Edward Witten. "(2+1)-dimensional gravity as an exactly soluble system". In: *Nuclear Physics B* 311.1 (1988), pp. 46–78. DOI: 10.1016/0550-3213(88)90143-5.

[15] Edward Witten. "Quantization of Chern-Simons Gauge Theory With Complex Gauge Group". In: *Commun. Math. Phys.* 137 (1991), pp. 29–66. DOI: 10.1007/BF02099116.