Free Actions on Spheres

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I pictured myself in a Denver bar that night, with all the gang, and in their eyes I would be strange and ragged and like the Prophet who has walked across the land to bring the dark Word, and the only Word I had was "Wow!"

On the Road, Jack Kerouac

To all people I have met in Cambridge.

Abstract

The problem of determining which finite groups can act freely on a sphere was first addressed by P. A. Smith in 1944, and resolved by I. Madsen, C. B. Thomas and C. T. C. Wall thirty years later. In this essay we show the negative part of the answer, namely that groups $\mathbb{Z}/p \oplus \mathbb{Z}/p$ and dihedral groups D_p cannot act freely on a sphere for any prime p. We also summarise T. Petrie's construction of a free action on S^{2q-1} of a non-abelian group of order pq, with p, q odd primes.

1 Introduction

Throughout this essay, we will be occupied with the following

Problem. For which finite groups G there is a positive integer n and a free action of G on the n-dimensional sphere S^n?

Here, each element of G acts on S^n via a homeomorphism. An action is *free* if non-trivial elements of G act without fixed points. In such a case

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we can form a quotient space S^n/G, and the map $S^n \to S^n/G$ is a covering. Therefore we can equivalently phrase our problem as: which finite groups occur as fundamental groups of manifolds whose universal cover is a sphere?

The class of all groups satisfying our question is easily seen to be closed under taking subgroups. In particular, we can try to phrase the answer in terms of forbidden subgroups of G.

The first result in this direction appeared in the paper of P. A. Smith [12] in 1944. Using methods that later gained the name of Smith theory, he proved that if f and g are continuous transformations of the sphere such that $f^p = g^p = \mathrm{id}$ and their fixed point sets are equal, then g is a power of p. In particular, for fixed-point-free transformations, we obtain that the group $\mathbb{Z}/p \oplus \mathbb{Z}/p$ cannot act freely on a sphere.

Stating the same observation more generally, if G acts freely on a sphere, then each of its subgroups of order p^2 is cyclic. If this is true, we say that G satisfies the p^2-condition. In 1957 J. Milnor [8] realised and proved that G also has to satisfy analogously defined $2p$-condition. Additionally, pq-condition (each subgroup of order pq is cyclic, for any primes p, q) turned out to be very important in the solution of the similarly stated problem of linear free actions on a sphere (which is the same as the problem of determining possible fundamental groups of Riemannian manifolds with constant positive curvature), and it was suspected that it might be important for the topological case. Then T. Petrie [11] in 1971 provided a construction of a free action of G on a sphere, where G is a non-cyclic group of order pq. Shortly thereafter Madsen, Thomas and Wall [7] proved that in fact satisfying p^2-condition and $2p$-condition is enough for G to act freely on a sphere.

We structure the essay as follows. First, we will shortly introduce the background in topology and representation theory used later. In chapter 3 we try to build our intuition on examples in dimensions up to 3. In chapters 4 and 5 we give restrictions on groups G acting freely on a sphere. In chapter 4 we prove that G does not contain $\mathbb{Z}/p \oplus \mathbb{Z}/p$ as a subgroup, using group cohomology and Thom's isomorphism theorem. In the following section we use Ronnie Lee's semicharacteristic invariant to rule out G containing D_p as a subgroup; this approach uses ideas from representation theory and cobordism theory. Finally, in chapter 6 we summarise Petrie's construction of a free action of $\mathbb{Z}/q \ltimes \mathbb{Z}/p$ on a sphere, although we prove that a linear free action does not exist in this case. For that, we introduce and use some facts about Brieskorn varieties.

2 Preliminaries

2.1 Notations

We write $\mathbb{Z}/n$ for the cyclic group of order n, and D_n for the dihedral group of order $2n$. The neutral element of a group is denoted as 1. If $g_1, \ldots, g_n \in G$, then $\langle g_1, \ldots, g_n \rangle$ denotes the subgroup of G generated by g_i-s. An element of order 2 is called an *involution*.

If $G \times X \to X$ is a group action (for X a set, a topological space, a vector space, ...) then we write $g.x$ to mean the image of x under the action of g. Throughout the essay, every group action is a left action.

Vectors, if lying in $\mathbb{R}^n$ or $\mathbb{C}^n$, will be usually denoted with bold letters, using the convention that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The sphere S^n is the subset of $\mathbb{R}^{n+1}$ of vectors $\mathbf{x}$ with $||\mathbf{x}|| = 1$. Similarly, the disk D^{n+1} consists of those $\mathbf{x}$ such that $||\mathbf{x}|| \leqslant 1$.

2.2 Actions on topological spaces

Let G be a finite group and X a normal topological space. An action of G on X is a homomorphism from G to the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X. We say that the action is free if every non-trivial element g of G acts on X without fixed points.

We can form a quotient space $X/\sim$, where the equivalence relation on X is given by $x_1 \sim x_2$ whenever $x_2 = g.x_1$ for some $g \in G$. We denote this space by X/G. The map $\pi : X \to X/G$ is a covering. Indeed, if y is in X/G and $x_1, \ldots, x_n$ are the points of $\pi^{-1}(y)$, then we can find neighbourhoods U_i for x_i with pairwise disjoint closures. Then if we define $V = \bigcap_{i=1}^{n} \pi(U_i)$, the preimage $\pi^{-1}(V)$ consists of n disjoint open subsets homeomorphic to V as claimed.

Here we have substantially used the assumption that G is finite. Indeed, S^1 as a group acts on itself with one orbit, but the map $S^1 \to \{pt\}$ is not a covering. It is still, however, a fibre bundle. The whole theory can be built more generally for any compact Lie group (for details see e.g. [1]); nevertheless, we will only consider finite groups throughout the essay.

We will be primarily interested in the case when X is a manifold. Then X/G inherits the manifold structure from X. If X is simply-connected, then the fundamental group $\pi_1(X/G)$ is isomorphic to G; we can reconstruct X from X/G by taking the universal cover. We can therefore analyse the situation from the point of view of X/G.

A map $\pi : P \to B$ is called a *principal G-bundle* over B, if G acts freely and transitively on each fibre, and π satisfies the local triviality condition (i.e. for finite G it is a covering). The space B is called the *base*, and P is the *total space* of the bundle. If $\pi : P \to B$ is a G-bundle and $f : A \to B$ is any continuous map, we can form a *pullback bundle*, whose total space is

$$A \times_B P := \{(a, p) \in A \times P \mid f(a) = \pi(p)\},$$

and the map to A is given by the projection.

Suppose EG is a contractible space on which G acts freely, and let $BG = EG/G$. The map $EG \to BG$ is called the *universal bundle* of G. The following classical theorem holds.

Theorem 2.1. If B is paracompact, then principal G-bundles over B are in bijection with maps $f : B \to BG$ up to homotopy. The correspondence going to the left is given by pulling back the universal bundle along f.

2.3 Homology and cohomology

Whenever G acts on a topological space X, we also obtain actions of G on $H_i(X)$ and $H^i(X)$ for each i, by functoriality of (co)homology. This way $H_i(X)$ and $H^i(X)$ may be considered as $\mathbb{Z}G$-modules or G-representations, depending on the coefficient ring and context.

The (co)homology should be understood as singular, apart from Chapter 5, where we need groups $C^i(X)$ to be finitely generated, and so cellular cohomology should be used. Most spaces we consider are compact manifolds, and every manifold of dimension not equal to four has a structure of CW-complex ([5][p. 529]). Fortunately, we will shortly see that even dimensions, in particular four, are not interesting for us, and so we can assume that every manifold we encounter has CW-structure (it is worth pointing out that the classifying space BG of a finite group G also can be equipped with a CW-structure). If X has a free G-action and X/G has CW-structure, then we can pullback such a structure to X. The group G then acts on the set of cells. If a non-trivial element of G fixed a cell in this action, then it would also fix its closure, and thus we would get a map $D^n \to D^n$. By Brouwer's theorem, such a map would have a fixed point, and we obtain a contradiction. Hence G acts freely on the set of cells of X.

2.4 Group cohomology

There are many equivalent definitions of group cohomology; as we adopt a topological point of view, we will prefer the following.

Definition 2.1. Let G be a finite group, then its cohomology groups with integral coefficients are

$$H^i(G; \mathbb{Z}) := H^i(BG; \mathbb{Z}).$$

All choices for BG are homotopy equivalent (e.g. [5][Prop 4.30, p.366]), so the right-hand side depends only on G. We will also need to use the values of $H^i(\mathbb{Z}/n; \mathbb{Z})$; admittedly the most efficient ways to compute them are using other definitions, but we will skip the proof anyway.

Theorem 2.2 ([16][p. 168])**.** We have

$$H^i(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z},\ i = 0, \\ \mathbb{Z}/n,\ i > 0 \text{ even}, \\ 0,\ i \text{ odd}. \end{cases}$$

2.5 Cobordism

Let X be a topological space. A (closed n-dimensional) singular manifold of X is a pair (M, f) where M is a closed n-dimensional manifold, and $f : M \to X$ is a continuous map. Singular manifolds (M_1, f_1) and (M_2, f_2) are called *bordant* if there exists a compact manifold W with boundary $\partial W = M_1 \cup M_2$, and a continuous map $F : W \to X$ such that $W|_{M_i} = f_i$, $i = 1, 2$. This is an equivalence relation; let $\Omega_n^O(X)$ be the set of its equivalence classes. The disjoint union makes $\Omega_n^O(X)$ a group, called the *n-th unoriented bordism group of X* [4][p. 585].

Each non-trivial element of $\Omega_n^O(X)$ has order 2; indeed for any pair (M, f) we may take $W = M \times [0, 1]$ and $F(m, t) = f(m)$. Similarly we can see that if f and g are homotopic maps from M to X, then (M, f) and (M, g) represent the same element in $\Omega_n^O(X)$. Since we will be interested mainly in G-actions, by Theorem 2.1 we would especially like to consider $\Omega_n^O(BG)$.

There is a connection between unoriented bordism and Whitney-Stiefel classes which we will use. Suppose ξ is a vector bundle with total space E and base space B.

Definition 2.2 ([9][Chapter 4])**.** For each k, the k-th Whitney-Stiefel class of ξ is an element $w_k(\xi) \in H^k(B; \mathbb{Z}/2)$, which satisfies the following axioms:

1. $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}/2)$; if ξ is n-plane bundle, then $w_k(\xi) = 0$ for all $k > n$;

2. if $f : A \to B$ is continuous, the Whitney-Stiefel classes of the pullback bundle are given by $w_k(f^*\xi) = f^*(w_k(\xi))$;

3. if ξ, η are two bundles over B, then

$$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \smile w_j(\eta);$$

4. if γ_1^1 is the canonical line bundle over $\mathbb{P}^1$, then $w_1(\gamma_1^1)$ is non-zero.

This is not a real definition, as it does not explain how such a choice should be made or why is it unique; for proof that Whitney-Stiefel classes are well-defined and other facts about them, see [9].

If I is a multiset of positive integers $\{i_1, \ldots, i_m\}$, with $i_1 + \ldots + i_m = k$, we say that I is a partition of k. We can then define the Whitney-Stiefel class $w_I(\xi) \in H^k(B; \mathbb{Z}/2)$ as the product $w_{i_1}(\xi) \cdot \ldots \cdot w_{i_m}(\xi)$. Crucially for our main result in Chapter 5, the following theorem lets us understand bordism groups $\Omega_n^O(X)$.

Theorem 2.3. [14][p. 108] Let X be a topological space and (M, f) be an n-dimensional singular manifold of X. Then the class of (M, f) in $\Omega_n^O(X)$ is uniquely determined by the collection of numbers

$$\langle w_I(TM) \smile f^*(x), [M] \rangle$$

where I is a partition of k, TM is the tangent bundle of M, and x is an element of $H^{n-k}(X; \mathbb{Z}/2)$. By uniquely determined we mean that two singular manifolds M_1, M_2 represent the same class in $\Omega_n^O(X)$ if and only if the numbers of M_1 and M_2 agree for each I and x.

2.6 Representation theory

Let G be a finite group, and $\mathbb{K}$ a field. A representation of G over $\mathbb{K}$ is a (finite-dimensional)[1] vector space V together with a homomorphism ρ from G to $\operatorname{End} V$, the group of invertible endomorphisms of V. If we have chosen a basis for V, we can also think about $\operatorname{End} V$ as the group $\mathrm{GL}_n(\mathbb{K})$ of invertible $n \times n$ matrices over $\mathbb{K}$, where n is the dimension of V. We may simply say that a representation is an action of a group on a vector space.

Among important examples of representations we have the *trivial representation* given by $\mathbb{K}$ with a trivial G-action, and the regular representation $\mathbb{K}G$, which is a $|G|$-dimensional vector space with basis indexed by elements of G, and G acting by $g.e_h = e_{gh}$. In fact, $\mathbb{K}G$ can be made into a $\mathbb{K}$-algebra

[1] This is not a part of the general definition, but it will be more convenient to work with this assumption.

if we define the multiplication as $e_g \cdot e_h = e_{gh}$ (we can similarly define group ring $\mathbb{Z}G$; the underlying structure is then a free abelian group on elements of G, instead of a vector space).

If V_1, V_2 are G-representations, then $V_1 \oplus V_2$ is too, with the action given by $g.(v_1, v_2) = (g.v_1, g.v_2)$. If V is a representation, then so is V^*, with the action $(g.\varphi)(v) = \varphi(g^{-1}.v)$.

If W is a subspace of V fixed by the action of G, we say that W is a subrepresentation of V. We say V is *reducible* if it has a subrepresentation different from $\{0\}$ and V. We say it is *decomposable* if it has subrepresentations V_1 and V_2 such that $V = V_1 \oplus V_2$. Finally, we call V *irreducible*, if it is not reducible, and *indecomposable*, if it is not decomposable.

Lemma 2.1 (Schur)**.** Let V, W be irreducible representations of G, and $f : V \to W$ a linear map such that $f(g.v) = g.f(v)$. Then f is the zero map or an isomorphism. Furthermore, if $\mathbb{K}$ is algebraically closed and $V = W$, then $f = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{K}$.

Proof. Observe that $\ker f$ is a subrepresentation of V, so f is either zero or injective. Similarly, $\operatorname{im} f$ is a subrepresentation of W, so f is either zero or surjective; in total it is zero or an isomorphism.

For the second part, let λ be an eigenvalue of f. Then $f - \lambda \cdot \mathrm{id}$ also commutes with the action of G but has a non-trivial kernel, and thus it is the zero map. □

In chapter 5 we will be particularly interested in invariant forms on representations. A bilinear form $\phi : V \times V \to \mathbb{K}$ is called G-invariant if $\phi(g.v, g.w) = \phi(v, w)$ for every $g \in G$ and $v, w \in V$.

Lemma 2.2. A non-zero G-invariant form on an **irreducible** representation V is always non-degenerate. If $\mathbb{K}$ is algebraically closed, then such a form is unique up to a scalar multiplication.

Proof. Let ϕ be that form; then the subspace

$$V^\perp = \{w \in V \mid \forall v \in V \ \phi(v, w) = 0\}$$

is a subrepresentation of V. Since ϕ is non-zero, $V^\perp$ is not the whole V, so it is trivial and ϕ is non-degenerate.

Let φ be another invariant form on V. Then ϕ defines a map from V to V^* by $v \mapsto (w \mapsto \phi(v, w))$. This is an isomorphism since ϕ is non-degenerate. A similar map can be defined for φ. By Schur's lemma, these two maps are scalar multiples of each other, but this implies that ϕ and φ also are. □

During the proof we have observed that the existence of a G-invariant form on V implies $V \simeq V^*$; the converse also turns out to be true. In particular, not every irreducible representation over $\mathbb{K}$ has an invariant form, but for the real and the complex cases, the situation is quite pleasant.

Lemma 2.3. Let V be a representation of a finite group G over $\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$, then there is a positive-definite symmetric form ϕ which is invariant on V. Similarly if $\mathbb{K} = \mathbb{C}$, then there is a positive-definite Hermitian form ϕ invariant on V.

Proof. Let $\langle \cdot, \cdot \rangle$ be any positive-definite symmetric/Hermitian form on V. Define

$$\phi(v, w) := \frac{1}{|G|} \sum_{g \in G} \langle g.v, g.w \rangle.$$

This is also a positive-definite symmetric/Hermitian form on V; moreover

$$\phi(h.v, h.w) = \frac{1}{|G|} \sum_{g \in G} \langle (gh).v, (gh).w \rangle = \frac{1}{|G|} \sum_{gh \in G} \langle (gh).v, (gh).w \rangle = \phi(v, w),$$

as $g \mapsto gh$ is a bijection on G. $\square$

Lemma 2.3 also implies that every representation of a finite group over $\mathbb{R}$ or $\mathbb{C}$ can, without loss of generality, consist only of actions by isometries. In particular, any such representation gives rise to an action of G on the sphere $\phi(v, v) = 1$.

Remark. Not every representation over $\mathbb{C}$ is self-dual (and Hermitian forms are not bilinear).

Corollary 2.1. Every reducible representation V over $\mathbb{R}$ or $\mathbb{C}$ is decomposable.

Proof. Let W be a proper subrepresentation of V and ϕ be a G-invariant positive-definite form on V. Define

$$W^{\perp} := \{v \in V \mid \forall w \in W \ \phi(v, w) = 0\}.$$

Now $W^{\perp}$ is a subrepresentation of V and $V = W \oplus W^{\perp}$, so V is decomposable as claimed. $\square$

In fact, one can similarly prove the following.

Theorem 2.4 (Maschke)**.** If the order of G is invertible in $\mathbb{K}$, then every reducible representation of G over $\mathbb{K}$ is decomposable.

Proof. See [3][p. 33]. □

In particular, if $\mathbb{K}$ is of characteristic zero, everything works fine. However, the assumption of Maschke's theorem will not be true when we will work in characteristic 2 in chapter 5, and we will use filtrations instead. If W is a subrepresentation of V, then the quotient space V/W becomes a G-representation via $g.[v] = [g.v]$. By a standard induction, for every V we can find a filtration

$$0 = V_0 \leq V_1 \leq \ldots \leq V_{n-1} \leq V_n = V$$

such that each quotient V_{i+1}/V_i is irreducible. We call V_{i+1}/V_i composition factors of V. The next theorem shows that the multiset of composition factors does not depend on the choice of the filtration.

Theorem 2.5 (Jordan-Hölder)**.** Let V be a representation of G, with

$$0 = V_0 \leq V_1 \leq \ldots \leq V_{n-1} \leq V_n = V$$

and

$$0 = V'_0 \leq V'_1 \leq \ldots \leq V'_{m-1} \leq V'_m = V$$

two filtrations such that $W_i = V_{i+1}/V_i$ and $W'_j = V'_{j+1}/V'_j$ are irreducible. Then $m = n$, and there is a permutation σ such that $W_i \simeq W'_{\sigma(i)}$.

Proof. See [3][p. 28]. □

3 Early observations

We are now ready to go back to our main problem. To find some groups G acting freely on S^n, we start with looking at small values of n.

3.1 The circle

The answer for the case of S^1 is simple.

Lemma 3.1. A finite group G has a free action on a circle if and only if it is cyclic.

Proof. If we let the generator of $\mathbb{Z}/n$ rotate S^1 by $\frac{2\pi}{n}$, the corresponding action is free.

Conversely, suppose a group G of order n acts freely on the circle, and let $\{A_1, \ldots, A_n\}$ be any orbit, with points listed in counterclockwise order. Let

$g \in G$ be such that $g.A_1 = A_2$. Then the interval $[A_1, A_2]$ does not contain any A_i-s in the interior, hence neither does $g.[A_1, A_2]$. Therefore $g.[A_1, A_2]$ has to have a form $[A_j, A_{j+1}]$ for some j. As $g.A_1 = A_2$, we get $j = 2$ or $j = 1$. In the latter case, the interval $[A_1, A_2]$ is mapped to itself via g, but such a map would have a fixed point, yielding a contradiction. Thus $j = 2$, so $g.A_2 = A_3$. By induction, $g.A_i = A_{i+1}$ for any i. Since G acts freely on each orbit, we need to have $G = \langle g \rangle$, which ends the proof. □

Alternatively, we can prove the same result using fundamental groups: S^1/G is a one-dimensional connected compact manifold, so has to be a circle, and then its n-fold coverings are in correspondence with subgroups of $\pi_1(S^1) \simeq \mathbb{Z}$ of index n.

3.2 Even-dimensional spheres

For S^2 the situation is described in the following, classical problem.

Lemma 3.2. If $f : S^n \to S^n$ is a continuous map with no fixed points, then f is homotopic to $-\operatorname{id}$. In particular, f preserves the orientation for odd n and reverses the orientation for even n.

Proof. We think of S^n as a subset of $\mathbb{R}^{n+1}$. Let us put

$$H(\mathbf{x}, t) := \frac{(1-t)f(\mathbf{x}) - t\mathbf{x}}{||(1-t)f(\mathbf{x}) - t\mathbf{x}||}.$$

This is well-defined and continuous because the denominator can be zero only if $\mathbf{x}$ and $f(\mathbf{x})$ are linearly dependent, which can only happen for $f(\mathbf{x}) = \mathbf{x}$ or $f(\mathbf{x}) = -\mathbf{x}$. The former case cannot happen, as f has no fixed points, and in the latter case, the denominator is equal to 1.

We see that $H(\mathbf{x}, 0) = f(\mathbf{x})$ and $H(\mathbf{x}, 1) = -\mathbf{x}$, so we have indeed constructed the homotopy. □

To determine whether f preserves the orientation or not, we look at its action on the top homology group $H_n(S^n; \mathbb{Z})$. It is isomorphic to $\mathbb{Z}$, so the action of G gives a homomorphism $G \to \operatorname{Aut} \mathbb{Z} = \{\pm 1\}$. If n is even, the kernel of this map is trivial by our lemma, and so G has at most two elements. This result can also be achieved quicker using Euler characteristics: for even n we see that

$$2 = \chi(S^n) = |G| \cdot \chi(S^n/G),$$

so $|G|$ divides 2.

If n is odd, then G preserves the orientation, which will turn out to be useful for us.

3.3 S^3

The unit sphere in $\mathbb{R}^4$ can be interpreted as the set of unit quaternions, which forms a group (incidentally isomorphic to SU(2)). Each finite subgroup of this group thus acts freely on S^3.

The classical way to characterise these subgroups is by observing that whenever q is a unit quaternion, then $x \mapsto qxq^{-1}$ acts isometrically on the space of pure quaternions (i.e. quaternions of form $ai + bj + ck$). Hence we get a map from S^3 to SO(3); its kernel is $\{\pm 1\}$. Now if $H \leq \mathrm{SO}(3)$ is a finite subgroup, then H acts on S^2. This action is no longer free, but the map $S^2 \to S^2/H$ is a branched covering, and by Hurwitz-type formulas we can obtain that such H is either cyclic, dihedral or an isometry group of a platonic solid (see [3][pp. 44-45] for a sketch of the proof).

Now we can pull these subgroups back to S^3 obtaining, respectively, cyclic groups, quaternion groups Q_{4n}, and so-called binary tetrahedral, octahedral and icosahedral groups.

The groups we obtained this way act on S^3 linearly; the general topological situation gives greater freedom for our constructions, which also makes it harder to prove restrictive results in our problem. The next two chapters will be devoted to such constraints.

4 Periodic cohomology and $\mathbb{Z}/p \oplus \mathbb{Z}/p$

In this section we will prove that $\mathbb{Z}/p \oplus \mathbb{Z}/p$ cannot act freely on any sphere, thus providing us with our first negative condition. Our main result will be that any finite group G acting freely on S^n has cohomology of period $n+1$, i.e. $H^i(G;\mathbb{Z}) = H^{i+n+1}(G;\mathbb{Z})$ for all $i > 0$. Then we will calculate the cohomology of $\mathbb{Z}/p \oplus \mathbb{Z}/p$ and observe it is not periodic.

Suppose that G acts freely on S^n with n odd.

Definition 4.1. The infinite-dimensional sphere S^∞ is the set of sequences $(x_1, x_2, \ldots) \in \mathbb{R}^\infty$ that are eventually zero and satisfy $\sum_{i=1}^{\infty} x_i^2 = 1$. It has a metric inherited from ℓ^2, and thus a topology.

S^n can be embedded into S^∞ as $(x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots)$, and for the rest of this section we will identify S^n with this subset of S^∞. We can actually extend the action of G on S^n to the whole S^∞. We achieve this as follows:

First, we can extend the action from S^n to the whole $\mathbb{R}^{n+1}$ via

$$g.\mathbf{x} = \begin{cases} \mathbf{0}, \ \mathbf{x} = \mathbf{0}; \\ ||\mathbf{x}|| \cdot g.\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right), \ \mathbf{x} \neq \mathbf{0}. \end{cases}$$

This action preserves lengths of vectors and is a free action on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. Now we consider the product action of G on $\mathbb{R}^\infty = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \cdots$, which fixes $S^\infty \in \mathbb{R}^{n+1}$. Hence G acts on S^∞, and does so freely.

Since S^∞ is contractible [5][p.88, ex. 1B.3.], S^∞/G is a classifying space for G. Hence its cohomology can help us understand the cohomology of G.

Theorem 4.1. Let G be a finite group acting freely on S^n (with n an odd number[2]). Then $H^i(G;\mathbb{Z}) = H^{i+n+1}(G;\mathbb{Z})$ for all $i > 0$.

Proof. We noticed that we have an inclusion $S^n \hookrightarrow S^\infty$, which is compatible with the action of G. Therefore we also have an inclusion $S^n/G \hookrightarrow S^\infty/G$, so we can write down the long exact sequence of the pair $(S^\infty/G, S^n/G)$ in cohomology:

$$\ldots \to H^{i-1}(S^n/G) \to H^i(S^\infty/G, S^n/G) \to H^i(S^\infty/G) \to H^i(S^n/G) \to \ldots$$

Since S^n/G is an n-manifold, its cohomology groups beyond the n-th are trivial, so we get an isomorphism $H^i(S^\infty/G, S^n/G) \simeq H^i(S^\infty/G)$ for $i > n+1$.

To calculate the cohomology of the pair $(S^\infty/G, S^n/G)$, let E be the subspace of S^∞ consisting of sequences of the form $(0, 0, \ldots, 0, x_{n+2}, x_{n+3}, \ldots)$. Of course, E is homeomorphic to the whole S^∞. At the same time, each point $\mathbf{p} \in S^\infty$ can be uniquely written as $\mathbf{p} = \alpha\mathbf{x} + \beta\mathbf{y}$ for $\alpha, \beta \in [0,1]$, $\alpha^2 + \beta^2 = 1$, $\mathbf{x} \in E$, $\mathbf{y} \in S^n$. Let $\alpha(\mathbf{p})$ mean the value of the coefficient α for the point $\mathbf{p}$.

Now let L be the set of points $\mathbf{p}$ such that $\alpha(\mathbf{p}) \leqslant \frac{1}{2}$, R be the set of points for which $\alpha(\mathbf{p}) \geqslant \frac{1}{2}$, and finally let M be the intersection of L and R, i.e. M contains $\mathbf{p}$ iff $\alpha(\mathbf{p}) = \frac{1}{2}$. We can imagine S^n being on the left of the picture and E on the right; then L is the "left half" of S^∞, R is the "right half", and M is the middle slice. We claim we can contract L to S^n.

Define a homotopy

$$H(\alpha\mathbf{x} + \beta\mathbf{y}, t) = c(\alpha, t)\,\mathbf{x} + \sqrt{1 - c(\alpha, t)^2}\,\mathbf{y},$$

[2]We know by Lemma 3.2 this is the interesting case, but we need this assumption – we saw that $\mathbb{Z}/2$ acts freely on **any** sphere, and its cohomology is not constant. Indeed, we will see we use the orientation in the proof.

where

$$c(\alpha, t) = \max(0, \alpha - t + t\alpha)$$

is a homotopy between id and $\max(0, 2\alpha - 1)$ on $[0, 1]$.

For $t = 0$ we have $H(\mathbf{p}, t) = \mathbf{p}$, and for $t = 1$ we see that $H(\cdot, 1)$ sends the whole L inside S^n, however its image is the whole S^∞. Moreover, this homotopy commutes with the action of G on S^∞, so we get an induced homotopy between pairs $(S^\infty/G, L/G)$ and $(S^\infty/G, S^n/G)$. Hence

$$H^i(S^\infty/G, S^n/G) = H^i(S^\infty/G, L/G).$$

Using the excision theorem for $\operatorname{int} L/G$ we obtain

$$H^i(S^\infty/G, S^n/G) = H^i(S^\infty/G, L/G) = H^i(R/G, M/G).$$

Let $B := E/G$. There is a map $\varphi : R \to E$ given by $\alpha\mathbf{x}+\beta\mathbf{y} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}$. Each fibre corresponds to all possible choices of $\beta\mathbf{y}$, for $\beta^2 \leqslant \frac{3}{4}$ and $\mathbf{y} \in S^n$, so to an $(n+1)$-dimensional disk D^{n+1}. Clearly, $\varphi : R \to E$ is a trivial disk bundle. Since φ is G-equivariant, we get an induced map $\pi : R/G \to B = E/G$, with disks as fibres.

We claim this map is an oriented disk bundle. Indeed, the quotient map $E \to B$ is a covering, so every point of B has a neighbourhood whose preimage in E has $|G|$ disjoint components. Then we can choose one of these components, and the trivialization of π on this neighbourhood comes from the trivialization of $\varphi : R \to E$. Transition maps between different trivializations are given by the action G on D^{n+1} (it will be the same action as the action of G on $\mathbb{R}^{n+1}$, which we have described before, restricted to D^{n+1}), which preserves the orientation (by Lemma 3.2), so the bundle is indeed oriented.

We wanted to compute $H^i(R/G, M/G)$. But R/G is the total space of our disk bundle $\pi : R/G \to B$, and M/G is the total space of the corresponding sphere bundle. By Thom's isomorphism theorem, we obtain

$$H^i(R/G, M/G) \simeq H^{i-n-1}(B).$$

Altogether,

$$H^i(S^\infty/G) \simeq H^{i-n-1}(B)$$

for $i > n + 1$. But B is E/G, and we observed E to be homeomorphic to S^∞. So S^∞/G is homeomorphic to B, and both are classifying spaces for G. In total we get

$$H^{i+n+1}(G; \mathbb{Z}) \simeq H^i(G; \mathbb{Z})$$

for $i > 0$, which concludes the proof. □

To calculate the cohomology of $\mathbb{Z}/p \oplus \mathbb{Z}/p$, we use the Künneth formula. Suppose G and H are groups whose classifying spaces we know. Then the space $EG \times EH$ is contractible, and $G \times H$ acts freely on it with quotient $BG \times BH$. Recall (e.g. [16][p. 87])

Theorem 4.2 (Künneth)**.** For (well-behaved) topological spaces X, Y we have a short exact sequence

$$0 \to \bigoplus_{i+j=m} H^i(X;\mathbb{Z}) \otimes H^j(Y;\mathbb{Z}) \to H^m(X \times Y;\mathbb{Z}) \to$$
$$\to \bigoplus_{i+j=m+1} \operatorname{Tor}_1^{\mathbb{Z}}(H^i(X;\mathbb{Z}), H^j(Y;\mathbb{Z})) \to 0,$$

which splits (non-canonically).[3]

Here Tor_1 is the derived functor of $\otimes$, which over $\mathbb{Z}$ we can easily and explicitly compute. This, however, will not be needed here, as we want to use the theorem for $X = Y = B\,\mathbb{Z}/p$, and it will be enough to consider the case of $m = 2k$ for some positive k. Recall (Theorem 2.2)

$$H^i(\mathbb{Z}/p;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, \ i = 0, \\ \mathbb{Z}/p, \ i \text{ even}, \ i > 0, \\ 0, \ i \text{ odd}. \end{cases}$$

Therefore $i + j = 2k + 1$ means that one of i, j has to be odd, and so $H^i(\mathbb{Z}/p;\mathbb{Z})$ or $H^j(\mathbb{Z}/p;\mathbb{Z})$ is zero. Therefore the Tor part is zero and we get

$$H^{2k}(\mathbb{Z}/p \oplus \mathbb{Z}/p;\mathbb{Z}) \simeq \bigoplus_{i+j=k} H^{2i}(\mathbb{Z}/p;\mathbb{Z}) \otimes H^{2j}(\mathbb{Z}/p;\mathbb{Z}) = (\mathbb{Z}/p)^{k+1}$$

(as $\mathbb{Z}/p \otimes \mathbb{Z}/p \simeq \mathbb{Z} \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p$, for tensoring over $\mathbb{Z}$).

This clearly means that the cohomology of $\mathbb{Z}/p \oplus \mathbb{Z}/p$ cannot be periodic. (In fact, groups with periodic cohomology are precisely groups that do not contain $\mathbb{Z}/p \oplus \mathbb{Z}/p$ for any p, see [2][Theorem 11.6, p. 262]).

Corollary 4.1. The group $\mathbb{Z}/p \oplus \mathbb{Z}/p$ cannot act freely on a sphere.

[3]Note that the better-known version for homology has the direct sum over $i+j = m-1$ and not $i+j = m+1$ in the last term. Both versions are valid for general chain complexes; they are equivalent since the cohomology of a cochain complex C^n is the same as the homology of $C_n := C^{-n}$. The minus is the reason for the change from $m-1$ to $m+1$.

Therefore a general group acting on a sphere cannot contain $\mathbb{Z}/p \oplus \mathbb{Z}/p$ as a subgroup. It turns out this can be equivalently phrased as a condition on Sylow subgroups.

Theorem 4.3 (Theorem 5.3.2 in [15])**.** Suppose that a finite group G does not contain $\mathbb{Z}/p \oplus \mathbb{Z}/p$ as a subgroup for any prime p. Then for any odd prime q the Sylow q-subgroup of G is cyclic, and the 2-Sylow subgroup of G is either cyclic or generalized quaternionic.

This is quite a strong condition for G; for instance in the case when the 2-Sylow subgroup of G is cyclic, it turns out we can fit G into an exact sequence

$$0 \to \mathbb{Z}/m \to G \to \mathbb{Z}/n \to 0$$

for some m, n, i.e. G is an extension of a cyclic group by another cyclic group (such groups are called *metacyclic*; we will work with one example of them in chapter 6). This property is a theorem by Burnside; its proof can be found in [15][Chapter 5].

5 Even representations and dihedral groups

J. Milnor in [8] was the first to point out that satisfying all p^2-conditions is not enough for G to act freely on a sphere. He showed

Theorem 5.1 (Milnor)**.** Let $T : S^n \to S^n$ be a map of order two with no fixed points. Then for every $f : S^n \to S^n$ with odd degree there exists $x \in S^n$ such that $f \circ T(x) = T \circ f(x)$.

In particular, if G acts freely on S^n, then every involution of G lies in its centre. Indeed, we have seen in Lemma 3.2 that each element of G acts on S^n with degree one, and $f \circ T(x) = T \circ f(x)$ implies $fT = Tf$ in G. Since we have ruled out G having a subgroup isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, there is at most one element of order 2.

For that, it will be enough to rule out D_p as a subgroup of G, for any odd prime p. Indeed, any two distinct involutions inside G generate either $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or a dihedral group D_m, which either contains D_p (for $p \mid m$ odd prime divisor) or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (for m being a power of two).

We will follow an approach of R. Lee from [6], which uses representation theory over a field of characteristic 2.

5.1 Grothendieck groups

In this subsection our aim is to define groups $R_e(G)$, which are target groups of Lee's invariant.[4]

Let $\mathbb{K}$ be a finite field of characteristic 2, and G a finite group. We define the group $R_{\mathbb{K}}(G)$ as follows.

Definition 5.1. $R_{\mathbb{K}}(G)$ is a free abelian group generated by symbols $[V]$, where V goes over all finite-dimensional representations of G over $\mathbb{K}$, quotiented out by the following relations:

- $[A] = [B]$ whenever $A \simeq B$;
- $[A] + [C] = [B]$ whenever we have a short exact sequence of representations $0 \to A \to B \to C \to 0$.

From the second axiom we get $[U \oplus V] = [U] + [V]$, hence we can interpret the addition in $R_{\mathbb{K}}(G)$ as taking the direct sum. There is a useful and fairly easy description of $R_{\mathbb{K}}(G)$ in terms of irreducible representations.

Lemma 5.1. $R_{\mathbb{K}}(G)$ is a free abelian group on $[V]$, where V goes over all irreducible representations of G.

Proof. We need to prove that any $[V]$ can be written as a linear combination of classes of irreducible representations, and that these classes are linearly independent. For the first part, we proceed by induction on the dimension of V; if $\dim V = 1$ then V is itself irreducible. Suppose $\dim V > 1$; if V is irreducible then again there is nothing to prove, so suppose V has a proper subrepresentation W. Then there is a short exact sequence $0 \to W \to V \to V/W \to 0$, hence $[V] = [W] + [V/W]$, and by induction $[W], [V/W]$ are linear combinations of classes of irreducible representations, so we are done.

In fact if

$$0 = W_0 \leq W_1 \leq W_2 \leq \ldots \leq W_n = V$$

is a filtration such that W_{k+1}/W_k is irreducible for each k, then

$$[V] = \sum_{k=0}^{n-1} [W_{k+1}/W_k]$$

by the above argument. This defines a map from $R_{\mathbb{K}}(G)$ to the free abelian group on classes of irreducible representations: it is well-defined on each

[4] Lee denotes these groups by $\widetilde{R}_{GL,ev}(G)$; I decided to simplify the convention.

$[V]$ by theorem 2.5 (Jordan-Hölder), and it preserves the relations defining $R_{\mathbb{K}}(G)$, since if $A \simeq B$ then A and B have the same composition factors (again by Jordan-Hölder), and if $0 \to A \to B \to C \to 0$ is a short exact sequence, then the multiset of composition factors of B is a sum of factors for A and C. This map is clearly injective and surjective, which finishes the argument. □

For our reasoning, we will be interested in a special class of representations.

Definition 5.2. Let V be a representation of G over $\mathbb{K}$. A symmetric form $\phi : V \times V \to \mathbb{K}$ is called *even*, if it is invariant under G, and for any involution $t \in G$ we have $\phi(v, t.v) = 0$ for all $v \in V$.

A representation V for which a non-degenerate even form exists is accordingly called *even*.

Remark. For any symmetric form ϕ and involution t, the map $v \mapsto \phi(v, t.v)$ is additive. Indeed,

$$\begin{aligned}\phi(v+w, t.(v+w)) - \phi(v, t.v) - \phi(w, t.w) &= \\ = \phi(v, t.w) + \phi(w, t.v) &= \phi(v, t.w) + \phi(t.w, t.(t.v)) = 0.\end{aligned}$$

Since $\phi(\lambda v, t.(\lambda v)) = \lambda^2 \phi(v, t.v)$, to check that ϕ is even it is enough to check $\phi(v, t.v) = 0$ on a basis.

The next lemma provides an intuition for the name.

Lemma 5.2. If G is a finite group of even order, then every even representation is even-dimensional.

Proof. Let V be a representation with an even form ϕ, and t be any involution of G. Define $\psi(v, w) := \phi(v, t.w)$. Now ψ is a non-degenerate bilinear form, and it satisfies $\psi(v, v) = 0$, so it is symplectic – and every vector space with a symplectic form is even-dimensional. □

Remark. Note if the order of G is odd, then it has no involutions, so every form is trivially even.

An important example of an even representation is given by the regular representation of G.

Lemma 5.3. For any G the regular representation $\mathbb{K}G$ is even.

Proof. Let e_g be a basis for $\mathbb{K}G$ with $g \in G$. We then define

$$\phi\left(\sum_g a_g g, \sum_g b_g g\right) = \sum_g a_g b_g.$$

This is quite easily seen to be bilinear and non-degenerate. Suppose t is an involution of G, and $\mathbf{v} \in \mathbb{K}G$. Then

$$\phi(\mathbf{v}, t.\mathbf{v}) = \sum_{g \in G} v_g v_{tg}.$$

But $t^2 g = g$, so the terms for g and tg will cancel out, as we work in characteristic 2. Hence ϕ is indeed even. $\square$

Another important source of even forms is given by intersection forms on even-dimensional manifolds. Let M be a $2n$-manifold (perhaps with boundary). Then the intersection form $\phi : H^n(M, \partial M; \mathbb{K}) \times H^n(M, \partial M; \mathbb{K}) \to \mathbb{K}$ is defined as $\phi(a, b) = \langle a \smile b, [M] \rangle$. Poincaré duals of a and b lie in $H_n(M; \mathbb{K})$; if they are given by submanifolds A and B which intersect transversally, then $A \cap B$ is a discrete set of points and its cardinality (modulo 2) is equal to $\phi(a, b)$.

Lemma 5.4. Let G be a group acting freely on a $2n$-dimensional manifold with boundary M. Then the intersection form $\phi : H^n(M, \partial M; \mathbb{K}) \times H^n(M, \partial M; \mathbb{K}) \to \mathbb{K}$ is even.

Proof. It is enough to check if the corresponding form on $H_n(M; \mathbb{K})$ is even. By abuse of notation, we will also call it ϕ. Let A be an n-submanifold of M. Then $[A] \in H_n(A; \mathbb{K})$ also represents an element in $H_n(M; \mathbb{K})$ via inclusion $A \hookrightarrow M$. If we perturb A, we can assume A intersects $t.A$ transversally (as transversality is a local property and t acts freely on M). If a point x lies in the intersection $A \cap t.A$, then so does $t.x$, and $t.x \neq x$ (since t acts without fixed points). Hence $A \cap t.A$ has an even number of points, so $\phi([A], t.[A]) = \phi([A], [t.A]) = 0$.

Since $H_n(M; \mathbb{K})$ is spanned by elements $[A]$ given by submanifolds, ϕ is indeed an even form. $\square$

Remark. $H^n(M, \partial M; \mathbb{K})$ does not have to be an even representation, as the intersection form does not have to be non-degenerate (e.g. when M is the surface of a cylinder).

Hyperbolic forms give other, more tautological examples of even forms.

Example. For any representation V, we can construct a *hyperbolic* form on $V \oplus V^*$ via

$$\phi : ((v_1, \varphi_1), (v_2, \varphi_2)) \mapsto \varphi_1(v_2) + \varphi_2(v_1),$$

where $v_i \in V$, $\varphi_i \in V^*$. This is an even form; indeed, for any nontrivial involution $t \in G$ we have

$$\phi((v, \varphi), t.(v, \varphi)) = \varphi(t.v) + (t.\varphi)(v) = \varphi(t.v) + \varphi(t^{-1}.v) = 2\varphi(t.v) = 0.$$

Remark. If V, W are even representations with forms ϕ_V, ϕ_W, then $V \oplus W$ is also an even representation, with the form ϕ defined by

$$\phi(v_1 + w_1, v_2 + w_2) := \phi_V(v_1, v_2) + \phi_W(w_1, w_2).$$

The next lemma allows us to produce even representations from any even form, not necessarily non-degenerate.

Lemma 5.5. Let ϕ be an even form defined on a representation V. We can define a map $\mathrm{Ad}_\phi : V \to V^*$ via

$$\mathrm{Ad}_\phi(v) = (w \mapsto \phi(v, w)) \in V^*.$$

Then the image of Ad_ϕ is an even representation.

Proof. If ϕ is non-degenerate, then Ad_ϕ is an isomorphism and V is an even representation, hence we are done.

If ϕ is degenerate, let $V^\perp$ be the kernel of Ad_ϕ. Then the image of Ad_ϕ is, by first isomorphism theorem, isomorphic to $V/V^\perp$. At the same time, we see that

$$V^\perp = \{v \in V : \phi(v, w) = 0 \ \forall w \in V\}.$$

Suppose that $\overline{v}, \overline{w} \in V/V^\perp$, and v, w are their respective lifts to V. If we put $\overline{\phi}(\overline{v}, \overline{w}) := \phi(v, w)$, then we see that $\overline{\phi}$ does not depend on the choice of the lift, and so is an even, non-degenerate form on $V/V^\perp$. This ends the proof. □

As we already saw that many representations we encounter are even, we want to be especially sensitive about non-even representations. The way to do it is to quotient out $R_{\mathbb{K}}(G)$ by all even representations.

Definition 5.3. The *relative even Grothendieck group* $R_e(G)$ is the quotient $R_{\mathbb{K}}(G)/E$, where E is the subgroup of $R_{\mathbb{K}}(G)$ generated by all $[V]$ for even representations V.

Remark. Lee's paper [6] uses the notation $R_{GL,ev}(G)$ for what we call $R_e(G)$; and additionally $\widetilde{R}_{GL,ev}(G)$ for the quotient of $R_e(G)$ by the class of the regular representation $[\mathbb{K}G]$. However, we know by Lemma 5.3 that $[\mathbb{K}G]$ represents zero already in $R_e(G)$.

It turns out we can easily describe this relative group in the fashion of Lemma 5.1.

Lemma 5.6. Let $V_1, \dots, V_k$ be all irreducible representations of G up to isomorphism. Suppose that W is an even representation, and write $[W] = \sum_{i=1}^k \alpha_i [V_i]$ (see Lemma 5.1). Then

- if $V_i^* \simeq V_j$, $i \neq j$, then $\alpha_i = \alpha_j$.
- if $V_i^* \simeq V_i$, and V_i is not an even representation, then α_i is even.

Conversely, if $\sum_{i=1}^k \alpha_i [V_i]$ is an element of $R_{\mathbb{K}}(G)$ with non-negative α_i-s satisfying the above conditions, then $\bigoplus_{i=1}^k \alpha_i V_i$ is an even representation representing this element.

Proof. For the second part, notice that the direct sum in question can be split up into several representations of form $V_i \oplus V_i^*$, which we know to be even by Example 5.1, and even irreducible representations V_j. As direct sums of even representations are even, we are done.

Now, for the first part, suppose W is a representation with an invariant non-degenerate even form ϕ. We will prove the claim by induction on the dimension of W. Let V be an irreducible subrepresentation of W (possibly $V = W$). If $\phi|_V$ is non-zero, then by Lemma 2.2 it is a non-degenerate form on V, and it is clearly even. Moreover,

$$V^\perp := \{w \in W \mid \phi(v, w) = 0 \ \forall v \in V\}$$

is a subrepresentation of W satisfying $W = V \oplus V^\perp$, and $\phi|_{V^\perp}$ is non-degenerate (if $w \in V^\perp$ annihilates ϕ on $V^\perp$, then it also does on $V^\perp \oplus V = W$), so $V^\perp$ is an even representation. We have $[W] = [V] + [V^\perp]$, the statement holds for $[V^\perp]$ by induction, and $[V]$ is a class of an irreducible even representation.

Suppose now $\phi|_V$ is zero, and define $V^\perp$ as above. Now we get $V \leq V^\perp$. We get an isomorphism $V^* \simeq W/V^\perp$ induced by ϕ. Indeed, the map

$$W/V^\perp \ni \overline{w} \mapsto (v \mapsto \phi(v, w)) \in V^*$$

is injective (from the definition of $V^\perp$), and if it were not surjective, then its image, considered as a subspace of V^*, would have a common non-zero element $v \in V$. But then v would annihilate ϕ both on $W/V^{\perp}$ and on $V^{\perp}$, which is impossible, since ϕ is non-degenerate.

Now the filtration $0 \leq V \leq V^{\perp} \leq W$ gives rise to

$$[W] = [V] + [V^{\perp}/V] + [W/V^{\perp}] = [V] + [V^{\perp}/V] + [V^*],$$

so if we show that $V^{\perp}/V$ is an even representation, we will be done by induction. But ϕ induces a well-defined bilinear form on $V^{\perp}/V$ which is clearly even and is also non-degenerate (as we have $\left(V^{\perp}\right)^{\perp} = V$ in finite-dimensional spaces). □

Corollary 5.1. $R_e(G)$ is isomorphic to $\mathbb{Z}^{k_1} \oplus (\mathbb{Z}/2)^{k_2}$, where k_1 is the number of dual pairs of irreducible representations of G, and k_2 is the number of irreducible representations of G that are self-dual, but not even (and the classes of the corresponding representations generate the whole group).

5.2 Representations of dihedral groups

We will use the theory developed in the previous section to dihedral groups D_p, with p an odd prime. In this short subsection, we determine the structure of $R_e(D_p)$. We will suppose that $\mathbb{K}$ contains a p-th root of unity ζ.

Let x be an element of order p, and y be an element of order 2 in D_p. Hence D_p has a presentation $\langle x, y \mid x^p = y^2 = (xy)^2 = 1\rangle$.

Since $\mathbb{K}^*$ is abelian, one-dimensional representations of D_p factor through the abelianization $D_p/[D_p, D_p]$, which is isomorphic to $\mathbb{Z}/2$. One-dimensional representations of $\mathbb{Z}/2$ are given by multiplication by λ such that $\lambda^2 = 1$; in characteristic two this means $(\lambda - 1)^2 = 0$, so $\lambda = 1$. Hence the trivial representation $\mathbb{K}$ is the only one-dimensional representation of D_p over $\mathbb{K}$.

We can also define two-dimensional representations of D_p, modelled on representations we know from the standard theory over $\mathbb{C}$.

Definition 5.4. For $i = 1, 2, \ldots, \frac{p-1}{2}$ let V_i be a representation of D_p given by

$$\rho(x) = \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{pmatrix}, \text{ and } \rho(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 5.2. Irreducible representations of D_p over $\mathbb{K}$ are the trivial one and V_i for $i = 1, 2, \ldots, \frac{p-1}{2}$.

Proof. Suppose V is such a representation. We can restrict V to a representation of $\langle x \rangle \simeq \mathbb{Z}/p$; this group is of order coprime to 2, so Maschke's theorem holds – every representation is completely reducible. We claim that every irreducible representation of $\mathbb{Z}/p$ is one-dimensional. Such a representation W is determined by the action of x, which is a matrix satisfying $\rho(x)^p = 1$. Let λ be its eigenvalue over $\overline{\mathbb{K}}$; then λ is a p-th root of unity, so is actually contained in $\mathbb{K}$. Now $\rho(x) - \lambda\,\mathrm{id} : W \to W$ commutes with x, and so is a map of representations. By Schur's lemma 2.1 we obtain $\rho(x) = \lambda \cdot \mathrm{id}$, which implies that W is irreducible if and only if it is one-dimensional.

Let us denote by W_j a one-dimensional representation of $\langle x \rangle$ where x acts by ζ^j, for $j = 0, 1, \ldots, p-1$. Hence V as a representation of $\mathbb{Z}/p$ decomposes as $\bigoplus \alpha_i W_i$ for some coefficients α_i.

Suppose that $\alpha_i > 0$ and $w \in W_i$ (so w is an eigenvector of $\rho(x)$ with eigenvalue ζ^i). Then

$$x.(y.w) = (xy).w = (xy)^{-1}.w = (yx^{-1}).w = y.(\zeta^{-i}w) = \zeta^{-i}y.w,$$

so $y.w$ is also an eigenvector of x. Therefore $\mathrm{span}(w, y.w)$ is fixed by x and by y, thus is a subrepresentation of V; as V is irreducible, we must have $V = \mathrm{span}(w, y.w)$. We already know all one-dimensional representations of D_p, so suppose w and $y.w$ are linearly independent and $\dim V = 2$. If $i = 0$, $w + y.w$ is fixed by both x and y, and its span is a one-dimensional subrepresentation, which would give us a contradiction. Hence $i \neq 0$. We can assume $i \in \{1, 2, \ldots, \frac{p-1}{2}\}$ by swapping w with $y.w$ if needed. Then writing the action of x and y in the basis $\{w, y.w\}$ we see that V is isomorphic to V_i, which ends our proof. □

Theorem 5.3. The group $R_e(D_p)$ is generated by $[\mathbb{K}]$ and $[V_i]$ for $i = 1, 2, \ldots, \frac{p-1}{2}$, all of order 2 and independent of each other.

Proof. By Lemma 5.6, it is enough to check that all these representations are self-dual, but not even. By Lemma 2.2, we just need to exhibit a non-zero invariant form that is not even (in our case $\mathbb{K}$ is not algebraically closed, but our representations stay irreducible when considered over $\overline{\mathbb{K}}$, so the conclusion of the lemma still holds). Clearly

$$\mathbb{K} \times \mathbb{K} \ni (a, b) \mapsto ab \in \mathbb{K}$$

is an invariant form on $[\mathbb{K}]$ that is not even. On V_i we can define

$$\varphi : V_i \times V_i \ni ((a, b), (c, d)) \mapsto ad + bc \in \mathbb{K}.$$

Now φ is clearly invariant under x and y, so is a D_p-invariant form on V_i; but

$$\varphi((a, b), y.(a, b)) = \varphi((a, b), (b, a)) = a^2 + b^2$$

is in general non-zero, so φ is not even. □

5.3 Semicharacteristics

Let M be a closed, oriented manifold of dimension $2n+1$, and G a finite group acting freely on M. Then G also acts on groups $H^i(M;\mathbb{K})$. These groups are formed from $C^i(M;\mathbb{K})$, which are finite-dimensional vector spaces over $\mathbb{K}$ (recall discussion in section 2.3); hence they are finite-dimensional themselves and thus represent elements in $R_{\mathbb{K}}(G)$.

Definition 5.5. The *semicharacteristic* of the pair (M, G) is the alternating sum

$$\chi_{1/2}(M;\mathbb{K}) = \sum_{i=0}^{n}(-1)^i[H^i(M;\mathbb{K})],$$

where the right-hand side is interpreted as an element in the relative Grothendieck group $R_e(G)$.

Clearly if M_1, M_2 are two manifolds with free G-action, then

$$\chi_{1/2}(M_1 \cup M_2;\mathbb{K}) = \chi_{1/2}(M_1;\mathbb{K}) + \chi_{1/2}(M_2;\mathbb{K}).$$

We will not actually need the following lemma, but it nicely demonstrates the methods we will use to prove Theorem 5.4.

Lemma 5.7. If M is a manifold with a free action of G, then

$$\chi_{1/2}(M;\mathbb{K}) = \chi_{1/2}(M;\mathbb{K})^*.$$

Proof. Consider $[\chi(M)]$ as an element of $R_e(G)$. On one hand,

$$\chi(M) = \sum_{i=0}^{2n+1}(-1)^i[C^i(M;\mathbb{K})],$$

and the action of G on $C^i(M;\mathbb{K})$ is free (recall section 2.3), therefore each $C^i(M;\mathbb{K})$ is a multiple of the regular representation $\mathbb{K}G$ and thus represents 0 in the relative group $R_e(G)$ (by Lemma 5.3).

On the other hand,

$$[\chi(M)] = \sum_{i=0}^{2n+1}(-1)^i[H^i(M;\mathbb{K})],$$

and by Poincaré duality and the universal coefficient theorem, we have $H^i(M) \simeq H_{2n+1-i}(M) \simeq H^{2n+1-i}(M)^*$, hence

$$0 = [\chi(M)] = \sum_{i=0}^{n}(-1)^i[H^i(M;\mathbb{K})] - \sum_{i=0}^{n}(-1)^i[H^i(M;\mathbb{K})^*] = \\ = \chi_{1/2}(M;\mathbb{K}) - \chi_{1/2}(M;\mathbb{K})^*.$$

□

The most important property of the semicharacteristic is given in the following theorem.

Theorem 5.4. Let W be a $(2n+2)$-manifold with boundary, on which G acts freely. Then

$$\chi_{1/2}(\partial W; \mathbb{K}) = 0.$$

Proof. Consider the long exact sequence of the pair $(W, \partial W)$ in cohomology.

$$0 \to H^0(W, \partial W) \to H^0(W) \to H^0(\partial W) \to \dots$$
$$\dots \to H^n(\partial W) \to H^{n+1}(W, \partial W) \xrightarrow{h} H^{n+1}(W) \to \dots$$

We can truncate this sequence to get $H^n(\partial W) \to H^{n+1}(W, \partial W) \to \operatorname{im} h \to 0$ at the end. Since this new sequence is exact, by taking an alternating sum we get

$$0 = [H^0(W, \partial W)] - [H^0(W)] + [H^0(\partial W)] - \dots$$
$$\dots + (-1)^{n+1}[H^{n+1}(W, \partial W)] - (-1)^{n+1}[\operatorname{im} h].$$

Using Lefschetz duality and the universal coefficient theorem, we get

$$H^k(W, \partial W) \simeq H_{2n+2-k}(W) \simeq H^{2n+2-k}(W)^*.$$

We also have $[V] = -[V]^*$ for any element of $R_e(G)$. Rewriting our equality, we thus get

$$\pm[\operatorname{im} h] = -[\chi(W)] + \chi_{1/2}(\partial W).$$

As $[\chi(W)] = 0$ by the same argument as in the proof of Lemma 5.7, it will be enough to prove $[\operatorname{im} h] = 0$.

Observe that the diagram

$$\begin{array}{ccc} H^{n+1}(W, \partial W) \times H^{n+1}(W, \partial W) & \xrightarrow{\phi} & \mathbb{K} \\ \downarrow{\scriptstyle h \times \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ H^{n+1}(W) \times H^{n+1}(W, \partial W) & \xrightarrow{\delta} & \mathbb{K} \end{array}$$

commutes, where ϕ is the intersection form on W, and δ comes from the Lefschetz duality. Indeed, the intersection form maps

$$H^{n+1}(W, \partial W) \times H^{n+1}(W, \partial W) \ni (\varphi, \psi) \mapsto (\varphi \smile \psi) \frown [W],$$

while δ maps (φ, ψ) to $\varphi(\psi \frown [W])$ (the map h on cocycle level is just the inclusion). By cup and cap product properties these are equal. Therefore also

$$\begin{array}{ccc} H^{n+1}(W,\partial W) & \xrightarrow{\mathrm{Ad}_\phi} & H^{n+1}(W,\partial W)^* \\ \downarrow h & & \downarrow \mathrm{id} \\ H^{n+1}(W) & \xrightarrow{D} & H^{n+1}(W,\partial W)^* \end{array}$$

commutes, where D is the Lefschetz duality (plus the universal coefficient theorem). Since D is an isomorphism we get $\operatorname{im} h \simeq \operatorname{im} \mathrm{Ad}_\phi$, which by Lemma 5.5 is an even representation. □

Theorem 5.4 allows us to use powerful tools from cobordism theory to study the semicharacteristics.

Definition 5.6. The *unoriented bordism group of free G-actions* in dimension d is denoted by $\mathcal{N}_d(G)$. Its elements are pairs (M, ρ), where M is a closed d-manifold and ρ is a free G-action on M, divided by an equivalence relation $(M_1, \rho_1) \sim (M_2, \rho_2)$ whenever there is a $(d+1)$-manifold W with boundary $\partial W = M_1 \cup M_2$, and a free G-action on W which agrees with ρ_i on M_i, for $i = 1, 2$.

The operation on $\mathcal{N}_d(G)$ is taking a disjoint sum. We allow taking M as an empty manifold; together with a (unique) G-action this will be the neutral element of our group.

Every element of $\mathcal{N}_d(G)$ has order (at most) two. Indeed, if (M, ρ) is a manifold with a free G-action, then $M \times [0, 1]$ has boundary $M \cup M$, and G acts on it via $g.(m, t) = (g.m, t)$.

There is also another description of the group $\mathcal{N}_d(G)$. We have stated in Theorem 2.1 that a free action of G on M induces a map $f : M/G \to BG$. Moreover, if $M_1 \cup M_2$ is the boundary of W, then $M_1/G \cup M_2/G$ is the boundary of W/G, and the map $W/G \to BG$ restricts to appropriate maps on M_1/G and M_2/G. Hence $\mathcal{N}_d(G)$ is the same as the group $\Omega^O_d(BG)$ we have introduced in section 2.5.

Using theorem 5.4, we can observe that the semicharacteristic is a well-defined map from $\mathcal{N}_{2n+1}(G)$ to $R_e(G)$.

Let H be a subgroup of G, and $i : H \to G$ be the inclusion map. We have obvious maps $i^* : N_d(G) \to N_d(H)$ and $R_e(G) \to R_e(H)$, as every G-action (on a manifold or a vector space) is also an H-action by restriction. These maps i^* commute with $\chi_{1/2}$ in the sense that the following diagram

$$\begin{array}{ccc} N_d(G) & \xrightarrow{i^*} & N_d(H) \\ \downarrow \chi_{1/2} & & \downarrow \chi_{1/2} \\ R_e(G) & \xrightarrow{i^*} & R_e(H) \end{array}$$

is commutative. We can also write maps i_* in the other direction. For $N_d(H)$, if M has a free H-action, then

$$\widetilde{M} := M \times_H G = \{(m,g) \in M \times G \mid (m,g) \sim (h.m, gh^{-1})\}$$

has a G-action by $k.[(m,g)] = [(m,kg)]$, which is free. Topologically, $\widetilde{M}$ is a disjoint union of $[G:H]$ copies of M.

Similarly, if V is a representation of H, then the induced representation of G is

$$\widetilde{V} := V \otimes_{\mathbb{K}H} \mathbb{K}G = \bigoplus_{g \in G} V/\sim,$$

where $\sim$ identifies v on g-th coordinate with $h.v$ on $\left(gh^{-1}\right)$-th coordinate. Thus as a vector space $\widetilde{V}$ is a direct sum of $[G:H]$ copies of V.

These two constructions are completely analogous, and so the following diagram is also commutative.

$$\begin{array}{ccc} N_d(H) & \xrightarrow{i_*} & N_d(G) \\ \big\downarrow \chi_{1/2} & & \big\downarrow \chi_{1/2} \\ R_e(H) & \xrightarrow{i_*} & R_e(G) \end{array}$$

It turns out that despite these similarities, the maps i^* and i_* we have constructed have different behaviour for bordisms and for representations, and we can utilise it to obtain strong results on the nature of free G-actions.

Lemma 5.8. The composition $\mathcal{N}_d(G) \xrightarrow{i^*} \mathcal{N}_d(H) \xrightarrow{i_*} \mathcal{N}_d(G)$ is the multiplication by $[G:H]$ (and so identity if $[G:H]$ is odd, and the zero map if the index is even).

Proof. We follow the proof by R. E. Stong ([13][p. 3]), using the connection between the bordism group and Whitney-Stiefel classes that we have introduced in Theorem 2.3.

Let $M \in \mathcal{N}_d(G)$. Write $X = M/G$ with G-action on M corresponding to a map $f : X \to BG$, and $Y = M/H$ with H-action corresponding to $f' : Y \to BH$. We have $[G:H]$-to-1 coverings $\pi' : BH \to BG$ and $\pi : Y \to X$. The diagram

$$\begin{array}{ccc} Y & \xrightarrow{f'} & BH \\ \big\downarrow \pi & & \big\downarrow \pi' \\ X & \xrightarrow{f} & BG \end{array}$$

is commutative, as the lower map is the same as the upper map quotiented by G on both sides. So $i_*(i^*(M))$ is given by a map $\pi' \circ f' = f \circ \pi : Y \to BG$. Now choose an element $\gamma \in H^k(BG; \mathbb{Z}/2)$, and let I be a partition of $d-k$. Then

$$\langle w_I(Y)\pi^*(f^*(\gamma)), [Y]\rangle = \langle \pi^*(w_I(X)f^*(\gamma)), [Y]\rangle = \\ = [G:H]\langle w_I(X)f^*(\gamma), [X]\rangle.$$

The first equality follows from the naturality of Whitney-Stiefel classes and the fact that the tangent bundle of Y is the pullback of the tangent bundle of X along π (as π is a covering, thus a local homeomorphism). The second equality follows from $\pi_*([Y]) = [G:H] \cdot [X]$ and the identity

$$h_*(\sigma) \frown \psi = h_*(\sigma \frown h^*(\psi))$$

that holds in general for the cap product.

Since all Whitney-Stiefel numbers agree, by Theorem 2.3 we get the desired bordism. □

We would like to take H to be the Sylow 2-subgroup of G so that $[G:H]$ is odd and H is as big as possible.

Lemma 5.9. If H is a non-trivial 2-group, the group $R_e(H)$ is of order two, generated by the class of the trivial representation $[\mathbb{K}]$.

Proof. Again, we follow the approach of [13][p. 4]. Let V be an irreducible representation of H, and consider it as a set. As a finite-dimensional vector space over a finite field with characteristic two, it has a finite number of elements, which is a power of 2. The zero vector has a one-element orbit, so there has to be another orbit with odd cardinality. Since H is a 2-group, such an orbit has also just one element and is formed by a vector v fixed by H. Now $\operatorname{span}(v)$ is a trivial subrepresentation, and V was irreducible, so $V = \operatorname{span}(v)$ is trivial.

Therefore a general representation W of H decomposes in $R_{\mathbb{K}}(H)$ as $\dim W \cdot [\mathbb{K}]$. As we have proved in Lemma 5.2 that each even representation is even-dimensional, and we know that $\mathbb{K}$ is self-dual (so $\mathbb{K} \oplus \mathbb{K}$ is even), we get the desired statement. □

Definition 5.7. The *Kervaire semicharacteristic* of a $(2n+1)$-manifold M is the sum

$$s_\chi(M) := \sum_{i=0}^{n} (-1)^i \dim H^i(M; \mathbb{Z}/2).$$

Theorem 5.5. Let M be a connected manifold with a free G-action, H be the Sylow 2-subgroup of G, and i_* the homomorphism $R_e(H) \to R_e(G)$. Then

$$\chi_{1/2}(M;\mathbb{K}) = s_\chi(M) i_*([\mathbb{K}]).$$

Proof. First of all, $H^i(M;\mathbb{K}) = H^i(M;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \mathbb{K}$, so we can use cohomology with coefficients in $\mathbb{K}$ instead of $\mathbb{Z}/2$ in the definition of Kervaire semicharacteristic. Now

$$\begin{aligned}\chi_{1/2}(M;\mathbb{K}) &= \chi_{1/2}(i_* i^* M;\mathbb{K}) = i_*(\chi_{1/2}(i^* M;\mathbb{K})) = \\ &= i_*\left(\sum_{i=0}^{n}(-1)^i [H^i(M;\mathbb{K})]\right) = i_*\left(\sum_{i=0}^{n}(-1)^i \dim H^i(M;\mathbb{K})[\mathbb{K}]\right) = \\ &= i_*\left(s_\chi(M)\cdot[\mathbb{K}]\right) = s_\chi(M) i_*([\mathbb{K}]).\end{aligned}$$

(Note: $[H^i(M;\mathbb{K})]$ and $[\mathbb{K}]$ are taken here with the action of H). □

Corollary 5.2. The dihedral group D_p cannot act freely on a sphere.

Proof. Suppose by contradiction that it acts freely on S^{2n+1} (we already know by Lemma 3.2 that even-dimensional spheres are of no interest). The only non-trivial cohomology up to degree n appears for $H^0(S^{2n+1};\mathbb{K})$; it corresponds to the (only) connected component of S^{2n+1}, hence it is one-dimensional and fixed by D_p. Thus $\chi_{1/2}(S^{2n+1};\mathbb{K}) = [\mathbb{K}]$.

At the same time, by Theorem 5.5, $\chi_{1/2}(S^{2n+1};\mathbb{K}) = s_\chi(S^{2n+1}) i_*([\mathbb{K}]) = i_*([\mathbb{K}])$.

Recall that irreducible representations of D_p over $\mathbb{K}$ are the trivial representation and two-dimensional representations V_i, $i = 1, 2, \ldots, \frac{p-1}{2}$. We claim

$$i_*([\mathbb{K}]) = [\mathbb{K}] + \sum_{i=1}^{\frac{p-1}{2}} [V_i].$$

When we restrict $i_*([\mathbb{K}])$ to a representation of $\mathbb{Z}/p$, we get $\mathbb{K}^p$ with $\mathbb{Z}/p$ acting by cyclic shifts of coordinates. The generator of $\mathbb{Z}/p$ then acts with eigenvalues $1, \zeta, \zeta^2, \ldots, \zeta^{p-1}$, where ζ is a p-th root of unity. But if ζ^i appears as an eigenvalue, then V_i has to appear as a subrepresentation; and similarly, eigenvalue 1 implies a trivial subrepresentation, so our formula for $i_*([\mathbb{K}])$ is true.

But clearly $i_*([\mathbb{K}]) \neq [\mathbb{K}]$ in $R_e(D_p)$ by Theorem 5.3 and we get a contradiction. □

6 Metacyclic groups and Brieskorn varieties

Let us go back to our main problem: suppose that G acts freely on a sphere. In chapters 4 and 5 we gave two restrictions on the structure of G in terms of its subgroups of order p^2 and $2p$, for p any prime. It is tempting to go one step further and ask about subgroups of order pq, with $p > q$ distinct odd primes. There are at most two groups of this order; one is cyclic (and we know we cannot get a contradiction in this case), and another one arises as a semidirect product $\mathbb{Z}/q \ltimes \mathbb{Z}/p$ in case $q \mid p-1$. We will denote it as $Z_{p,q}$ for short. It can be presented as

$$Z_{p,q} = \langle x, y \mid x^p = y^q = 1,\ yxy^{-1} = x^\sigma \rangle,$$

where σ is a primitive q-th root of unity modulo p.

The following result is encouraging for our further consideration.

Lemma 6.1. The group $Z_{p,q}$ cannot act **linearly** and freely on any sphere.

Proof. Suppose that such an action exists, i.e. there is a (real or complex) representation $\rho : Z_{p,q} \to \operatorname{End} V$ such that each non-trivial element $g \in Z_{p,q}$ acts on $V \setminus \{0\}$ without fixed points. Let H be a nontrivial subgroup of $Z_{p,q}$, and choose a non-zero vector v. Then $\sum_{h \in H} h.v$ is fixed by any non-trivial element of H, and so must be equal to zero:

$$\sum_{h \in H} h.v = 0.$$

Let us sum these equalities for all cyclic subgroups of $Z_{p,q}$. Any non-trivial element of $Z_{p,q}$ is in exactly one such subgroup (as p and q are primes), so we get

$$kv + \sum_{g \in Z_{p,q} \setminus 1} g.v = 0,$$

where k is the number of cyclic subgroups of $Z_{p,q}$. At the same time, we can write our equality for $H = G$.

$$v + \sum_{g \in Z_{p,q} \setminus 1} g.v = 0.$$

Comparing, we get $(k-1)v = 0$. But $k > 1$ (in fact, $k = p + 1$) and we are in characteristic zero, so we get a contradiction. $\square$

Surprisingly, it is possible for $Z_{p,q}$ to act non-linearly on a sphere. One of the first such examples was constructed by Petrie in [11]; we will summarise his argument in the rest of this section.

6.1 Brieskorn varieties

The main idea is to use so-called *Brieskorn varieties*, named after Egbert Brieskorn. Let $a_1, \ldots, a_{n+1}$ be positive integers, and consider the **complex** multi-variate polynomial

$$f(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \ldots + z_{n+1}^{a_{n+1}}.$$

Let $V := f^{-1}(0)$. The behaviour of V in $\mathbf{0}$ is usually singular, and we would like to understand it in terms of something non-singular. Choose a small $\varepsilon > 0$, and let S_ε be the sphere with radius ε around $\mathbf{0}$ (it has dimension $2n+1$).

Definition 6.1. The Brieskorn variety K is the intersection of V with the sphere S_ε. It is a smooth variety of dimension $2n-1$.

If $n = 1$ and $\gcd(a_1, a_2) = 1$, it can be shown that K is the torus knot corresponding to the pair (a_1, a_2). Brieskorn varieties can be therefore thought of as generalizations of these knots.

Topologically, K is "almost" a sphere. Namely, it is $(n-2)$-connected, so by Poincaré duality the homology groups $H_i(K;\mathbb{Z})$ can be non-zero only for $i = 0, n-1, n, 2n-1$. We will sketch the proof of this fact using Morse theory, following Milnor [10][Chapter 5].

Let

$$\phi : S_\varepsilon \setminus K \to S^1, \quad \phi(\mathbf{z}) = \frac{f(\mathbf{z})}{|f(\mathbf{z})|}.$$

The map ϕ turns out to be a fibre bundle; let F_θ denote the fibre over $e^{i\theta}$. Put $a(\mathbf{z}) := \log|f(\mathbf{z})|$, and let $a_\theta = a|_{F_\theta}$. The critical points of a are the same as those of f which are contained in $S_\varepsilon \setminus K$. By direct calculation, those are the points such that $\nabla \log f(\mathbf{z})$ is a complex multiple of the vector $\mathbf{z}$. If $\mathbf{z} \in F_\theta$ is such a point, then the tangent space $T_{\mathbf{z}}F_\theta$ is actually a complex vector space, as it is the ortogonal complement of $\operatorname{span}(\mathbf{z}, i\nabla \log f(\mathbf{z}))$ as a real space, and these two vectors are then linearly dependent over $\mathbb{C}$.

The quadratic form given by the Hessian of a_θ at a critical point $\mathbf{z}$ can be computed to be of the form

$$H(\mathbf{v}) = \operatorname{Re}\left(\sum b_{jk} v_j v_k\right) - c||\mathbf{v}||^2,$$

where b_{jk} are some complex numbers, and c is a positive real numbers.

The Morse index at a critical point is the maximal dimension of the subspace on which H is negative definite. The main observation is that whenever $H(\mathbf{v}) \geqslant 0$, then $iH(\mathbf{v}) < 0$. Indeed, the first part of our expression for $H(\mathbf{v})$ gets multiplied by -1 if we multiply $\mathbf{v}$ by i, so $H(\mathbf{v}) + H(i\mathbf{v}) = -2c||\mathbf{v}||^2 < 0$. Now if we split the tangent space at $\mathbf{z}$ as $T_- \oplus T_+$ (sum of real vector subspaces), with H negative definite on T_- and positive semi-definite on T_+, then by our observation iT_+ is a subspace of T_-. Therefore the dimension of T_- is at least half the dimension of $T_+ \oplus T_-$, which is $2n$. Hence every critical point is of index at least n.

By standard arguments of Morse theory we can, if needed, improve functions a_θ and a to functions s_θ and s, whose critical points are all non-degenerate, of index $\geqslant n$, and their number is finite. Now we can prove

Theorem 6.1. The manifold K is $(n-2)$-connected.

Proof. Let $N_\eta(K) := \{\mathbf{z} \in S_\varepsilon \mid |f(\mathbf{z})| \leqslant \eta\}$, where η is small enough so that $N_\eta(K)$ does not contain critical points of s. The function s is a Morse function defined on $S_\varepsilon \setminus \operatorname{int} N_\eta(K)$, so by the main theorem of Morse theory, S_ε can be built up from $N_\eta(K)$ by adjoining cells of dimension $\geqslant n$. Such additions do not change homotopy groups π_i up to the $(n-2)$-nd, so

$$\pi_i(N_\eta(K)) = \pi_i(S_\varepsilon) = 0, \; i = 1, 2, \ldots, n-2.$$

Now by general results of Łojasiewicz about real algebraic sets, K is a retract of $N_\eta(K)$ if η is sufficiently small. Thus $\pi_i(K)$ is trivial for $i = 1, 2, \ldots, n-2$. □

(With similar methods it can be proven that fibres F_θ have homotopy type of CW-complexes with dimension n (in particular, $H_n(F)$ is free abelian) and that they are parallelizable manifolds with boundary K).

As mentioned before, K thus has all homology groups $H_i(K)$ trivial except for $i = 0, n-1, n, 2n-1$. To compute $H_{n-1}(K)$ and $H_n(K)$, we may use Alexander duality and Poincaré duality to observe that

$$H_n(S_\varepsilon \setminus K) \simeq H^n(K) \simeq H_{n-1}(K).$$

Now $S_\varepsilon \setminus K$ is the total space of the fibration $\phi : S_\varepsilon \setminus K \to S^1$ we have investigated before. Fibre bundles over a circle have a Wang sequence, which in our case gives

$$0 \to H_n(K) \to H_n(F_0) \xrightarrow{h_* - \mathrm{id}} H_n(F_0) \to H_{n-1}(K) \to 0.$$

Here, $h : F_0 \to F_0$ can be constructed as $h_{2\pi}$, where $h_t : F_0 \to F_t$ is a continuous family of homeomorphism given by homotopy lifting property of the fibration. If $h_* - \mathrm{id}$ has determinant ± 1, then groups $H_n(K)$ and

$H_{n-1}(K)$ are trivial, and for $n \geqslant 3$ this is enough to conclude that K is a sphere.

The polynomial $\Delta(t) := \det(t \cdot \mathrm{id} - h_*)$ is a generalization of Alexander's polynomial of a knot. We can compute it using

Theorem 6.2 (Brieskorn-Pham)**.** Groups $H_n(F_0)$ are free abelian of rank $\prod_{i=1}^{n+1}(a_i - 1)$. The polynomial $\Delta(t)$ is given by

$$\Delta(t) = \prod (t - \omega_1 \cdots \omega_{n+1}),$$

where in the product each ω_i goes over all a_i-th roots of unity different from 1.

6.2 Petrie's method

In this subsection, denote the cyclic group of order n by C_n, and we will write $\mathbb{Z}/n$ whenever we are interested in the ring. Let us also write $Z_{p,q}$ as G for short, and let $\zeta_m := e^{2\pi i/m}$ be the m-th primitive root of unity. We have a short exact sequence

$$0 \to C_p \to G \to C_q \to 0,$$

where $C_p = \langle x \rangle$ and $C_q = \langle y \rangle$. The map $G \to C_q$ gives rise to a one-dimensional representation of G, where x acts trivially, and y acts by multiplication by ζ_q. Similarly, the map $C_p \to G$ gives us an induced q-dimensional representation of G, where y acts by a cyclic shift of coordinates, and x multiplies i-th coordinate by $\zeta_p^{\sigma^i}$ (recall σ is such that $yxy^{-1} = x^\sigma$).

Let V be the direct sum of these two representations; it has dimension $q+1$. Now let us define

$$f(\mathbf{z}) = z_1^p + z_2^p + \ldots + z_q^p + z_{q+1}^l,$$

where $l = q^k$, and k will be chosen later. This function is invariant under the action of G. Unfortunately, if we define K by $S_\varepsilon \cap f^{-1}(0)$, then the action of G on K will not be free. Instead, we want to choose a small η, and put $K = S_\varepsilon \cap f^{-1}(\eta)$. Suppose that a non-trivial $g \in G$ fixes a point of K; then so does g^i for any i, and any conjugate of g^i. We can check that each non-trivial element of $Z_{p,q}$ has a power that is conjugate to either x or y. Hence it is enough to check if x or y have any fixed points. We can easily check that the fixed space of x is of the form $(0, \ldots, 0, z)$, for y is of the form $(z, \ldots, z, 0)$, and that for almost all choices of ε and η, these points do not lie on K. Thus we can indeed choose (ε, η) so that G acts freely on K.

Lemma 6.2. In this setting, $\Delta(1)$ is a power of q.

Proof. Recall that for any number m and a complex number t, we have

$$\sum_{i=1}^{m-1}(t-\zeta_m^i)=\frac{t^m-1}{t-1}=1+t+\ldots+t^{m-1},$$

and this last part is true even for $t=1$ when the middle expression is not well-defined.

By Brieskorn-Pham Theorem, we need to calculate

$$\prod_{i_1,\ldots,i_q=1}^{p-1}\ \prod_{j=1}^{l-1}\left(1-\zeta_p^{i_1}\ldots\zeta_p^{i_q}\zeta_l^j\right)=\prod_{i_1,\ldots,i_q=1}^{p-1}\ \prod_{j=1}^{l-1}\left(1-\zeta_p^{i_1+\ldots+i_q}\zeta_l^j\right).$$

Let n_s be the number of q-tuples $(i_1,i_2,\ldots,i_q)\in(\mathbb{Z}/p)^*$ with sum congruent to s modulo p. From obvious action of $(\mathbb{Z}/p)^*$ we can conclude that $n_1=n_2=\ldots=n_{p-1}$. Observe that now our expression becomes

$$\prod_{j=1}^{l-1}\left(1-\zeta_l^j\right)^{n_0}\cdot\prod_{i=1}^{p-1}\prod_{j=1}^{l-1}\left(1-\zeta_p^i\zeta_l^j\right)^{n_1}.$$

We can use our observation from the beginning: The first product is equal to $\left(1^0+1^1+\ldots+1^{l-1}\right)^{n_0}=q^{n_0k}$. The second is equal to the n_1-th power of

$$\prod_{j=1}^{l-1}\left(1+\zeta_l^j+\zeta_l^{2j}+\ldots+\zeta_l^{(p-1)j}\right)=\prod_{j=1}^{l-1}\frac{\zeta_l^{pj}-1}{\zeta_l^j-1}.$$

But multiplication by p is a bijection on non-zero elements of $\mathbb{Z}/l$, so, in fact, each possible ζ_l^j-1 appears exactly once in the numerator and the denominator. Hence the second part of the product is equal to 1. □

Corollary 6.1. The group $H_q(K)$ is trivial, and $H_{q-1}(K)$ is annihilated by a power of q.

Proof. This follows from the exact sequence

$$0\to H_q(K)\to H_q(F)\xrightarrow{h_*-\mathrm{id}}H_q(F)\to H_{q-1}(K)\to 0$$

(with F as in the previous subsection) and the fact that $\det(h_*-\mathrm{id})=\pm\Delta(1)$, which implies that $h_*-\mathrm{id}$ is injective with cokernel of cardinality being a power of q. □

Petrie performs this calculation on the level of representations to deduce the $\mathbb{Z}G$-module structure on $H_{q-1}(K)$, which turns out to consist of two factors. The modulo l group ring $(\mathbb{Z}/l)C_q$ is a $\mathbb{Z}G$-module, on which x acts trivially, and y by action on C_q. Less trivially, let

$$\Gamma_l := (\mathbb{Z}/l)[t]/(1 + t + t^2 + \ldots + t^{p-1}),$$

with action given by $x.f(t) = t \cdot f(t)$ and $y.f(t) = f(t^\sigma)$. Note that indeed $x^\sigma.(y.f)(t) = y.(x.f)(t) = t^\sigma f(t^\sigma)$, as we would require.

Theorem 6.3. [11][Thm 2.4] As a $\mathbb{Z}G$-module

$$H_{q-1}(K) \simeq \frac{n_0}{q}(\mathbb{Z}/l)C_q \oplus n_1\Gamma_l,$$

where n_0, n_1 are as in the proof of Lemma 6.2.

From that, by homological and K-theoretical arguments one can choose $l = q^k$ so that $H_{q-1}(K)$ has a free resolution

$$0 \to F \xrightarrow{B} F \to H_{q-1}(K) \to 0,$$

where F is a free module over $\mathbb{Z}G$ with rank $2m$ for some m, and B is a matrix of form $B = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$, where A is some $m \times m$ matrix. Here A^* is defined by $a^*_{ij} = \overline{a_{ji}}$, where $\overline{a} = \sum_{g\in G} a_g g^{-1}$ for $a = \sum_{g\in G} a_g g$.

In surgery theory, there is a theorem of Wall, which we quote in an altered version used by Petrie.

Theorem 6.4 ([11][Thm 5.6])**.** A necessary and sufficient condition that surgery is possible on K/G yielding a manifold N with $\pi_1(N) = G$ and such that the universal cover Σ of N is a homotopy sphere, is that there exists a free module F of finite rank k over $\mathbb{Z}G$ and a $k \times k$ matrix B such that $B^* = -B$ and an exact sequence

$$0 \to F \xrightarrow{B} F \to H_{q-1}(K) \to 0,$$

such that the linking form (defined in Wall's paper)

$$\varphi : H_{q-1}(K) \to \mathrm{Hom}_{\mathbb{Z}G}(H_{q-1}(K), \mathbb{Q}G/\mathbb{Z}G)$$

is given by

$$\varphi(\mathbf{y})(\mathbf{x}) = \sum x_i B^{-T}_{ij} \overline{y_j} \pmod{\mathbb{Z}G}$$

when $\mathbf{x}, \mathbf{y} \in F = (\mathbb{Z}G)^{\oplus k}$.

Petrie manages to prove that, up to equivalence, there is a unique Hermitian form on $H_{q-1}(K)$, and so the matrix giving the linking form has to be the same matrix B we have produced earlier. Therefore the surgery is possible, and we get a homotopy sphere Σ of dimension $2q-1$ with a free action of the group G. Our Σ does not have to be homeomorphic to S^{2q-1}, but if it is not, then $\Sigma \# \Sigma$ is, and so G acts freely on S^{2q-1}.

7 Final remarks

In 1975 I. Madsen, C. B. Thomas and C. T. C. Wall gave a complete solution to our problem in the paper [7].

Theorem 7.1. A finite group G can act freely on a sphere if and only if all its subgroups of order p^2 and $2p$ (for p a prime) are cyclic. Moreover, the action can be assumed to be smooth for some differential structure on the sphere (which might be an exotic structure).

Therefore, the restrictions we have proved in chapters 4 and 5 are, in fact, also sufficient conditions for the existence of a free action. The proof of the theorem, however, is beyond the scope of this essay; it uses surgery and obstruction theory.

There are several variations of this problem one might also consider. We have already commented on finding linear free actions on a sphere. It is an interesting (although easier) problem; its full solution can be found in J. Wolf's book [15][Chapters 5,6,7]. For topological actions we can analyse free actions on products of several spheres; it is still an ongoing direction of research.

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