THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_n

Abstract. We introduce the local Langlands correspondence for GL_n and define all of the objects involved in the statement of the theorem. Emphasis is placed on thorough exposition and careful explanation of results encountered.

Contents

Introduction

The local Langlands correspondence for GL_n is a deep and powerful theorem which first arose as the result of various conjectures made by Langlands in a letter to Weil ([28]) in 1967, which was the genesis of the Langlands program. The eponymous result posits the existence of a bijection:

$$\left\{ \begin{array}{c} \text{Irreducible admissible} \\ \text{representations of} \\ GL_n(K) \end{array} \right\}_{/\cong} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Frobenius semisimple} \\ n\text{-dimensional complex} \\ \text{representations of } W_K \end{array} \right\}_{/\cong},$$

where K denotes a local field and W_K the corresponding Weil group. This bijection should also preserve certain salient properties called L and ϵ-factors, the twisting of characters and should reduce to local class field theory in the case where $n = 1$. This result has since been proven in all cases and has many far-reaching consequences within geometry and number theory.

The first case to be proved was when K is an archimedean local field (so $K = \mathbb{R}$ or $K = \mathbb{C}$). This was proved for $K = \mathbb{C}$ by Zhelobenko and Naimark ([48]) and later for $K = \mathbb{R}$ by Langlands ([29]) himself. These proofs have since been exposited in an article by Knapp ([25]).

If K is a non-archimedean local field of equal characteristic (so $K = \mathbb{F}_q((t))$ for some power of a prime q) then the result was proven by Laumon, Rapoport and Stuhler ([30]) who generalised the methods used by Drinfeld ([10]) in proving the case where $n = 2$ by making use of moduli spaces of $\mathcal{D}$-elliptic sheaves associated to a global function field.

In the final case where K is a non-archimedean local field of mixed characteristic (so a finite extension of $\mathbb{Q}_p$ for some prime p), the result was proven by Harris and Taylor ([17]) and then significantly shortened by Henniart ([19]) shortly after. Both of these proofs are global in nature and hinge on an understanding of the ℓ-adic cohomology of a certain family of Shimura varieties. More recently an alternative proof was developed by Scholze ([37]) which avoids Henniart's 'numerical local Langlands theorem' and instead makes use of a geometric result which describes the inertia-invariant nearby cycles in certain regular situations.

This essay serves to provide an introduction to the local Langlands correspondence for GL_n in the case of K being a local field of mixed characteristic, with the goal of defining all of the objects involved in the statement as well as motivating their involvement. We assume that the reader has a working familiarity with the theory of local fields, profinite groups, basic representation theory and local class field theory. Some possible references are [38], [46], [12] and [20], respectively.

We begin in section 1 by stating the main result as given by Scholze in [37] and explaining how it corresponds to local class field theory in the case where $n = 1$. We also briefly speak of some other properties which are preserved by the bijection, namely the conductor and depth.

In section 2 we will define everything involved in the GL_n side of the bijection. We begin by introducing the notion of admissible representations and some basic results concerning these. We then give a brief overview of the method of parabolic induction which we use to introduce supercuspidal representations, a complete description of which for $GL_n(K)$ was given by Bushnell and Kutzko in [6]. This allows us to state the Bernstein-Zelevinsky classification, which allows us to relate irreducible admissible representations to supercuspidal representations in a way we shall make precise. We then conclude by defining the L and ϵ-factors on this side of the bijection as well as some of their properties and how they may be defined inductively.

In section 3 we will define everything involved in the Galois side of the bijection. We begin by introducing the necessary background material required to define the Weil group and then move on to defining Weil-Deligne representations and discussing some of their basic properties. We then prove some important classical results such as Grothendieck's ℓ-adic monodromy theorem, which allow us to relate these representations to ℓ-adic representations. These representations are 'more natural' in some sense and are understood better, so relating the two is crucial to understanding this side of the bijection. To conclude we define the L and ϵ-factors on this side of the bijection and give a sample calculation.

Finally in section 4 we conclude the essay by discussing some further consequences of the correspondence such as the Satake isomorphism, as well as some recent results which have been postulated or proved in the area.

None of the results presented in this essay are original. Much of our work follows the order of exposition given by Carayol ([7]), Kudla ([26]) and Wedhorn ([45]), filling in any gaps and giving further details where appropriate. We place extra emphasis on careful explanation of the proofs of the results encountered.

Notation and Conventions

Throughout this essay we fix the following notation and conventions, unless stated otherwise.

- K is a local field of mixed characteristic, i.e $[K : \mathbb{Q}_p] < \infty$ for some prime p.
- $\mathcal{O}_K$ denotes the valuation ring of K.
- k is the residue field of K, which has cardinality q and characteristic p.
- ℓ denotes a prime number not equal to the residue characteristic p.
- π_K is a uniformiser for K.
- W_K denotes the Weil group of K.
- I_K denotes the inertia subgroup of W_K.
- P_K denotes the wild inertia subgroup of I_K, its unique maximal pro-p subgroup.
- G_K denotes the absolute Galois group of K.
- $\mathbb{N} = \{1, 2, \ldots\}$.
- $\mathrm{Rep}^{\mathrm{adm}}(G)$ denotes the set of admissible representations of G.
- I_n denotes the $n \times n$ identity matrix.

We shall also follow the following conventions related to characters:

- A *quasi-character* of G is a continuous homomorphism $\psi : G \to \mathbb{C}^\times$.
- A *unitary character* of G is a continuous homomorphism $\psi : G \to \{z \in \mathbb{C} : |z| = 1\}$.

1. Formulating the Correspondence

In this introductory section we shall state the eponymous local Langlands correspondence for GL_n. We begin by providing motivation for the statement by explaining how the case where $n = 1$ corresponds to local class field theory. For a detailed discussion of local class field theory, see [33] or [38]. Our exposition shall follow that of [45].

1.1. The Local Langlands Correspondence for GL_1.

Recall the main theorem of local class field theory:

Theorem 1.1. *(Local Artin Reciprocity) There exists a unique topological isomorphism:*

$$Art_K : K^\times \to W_K^{ab},$$

where W_K^{ab} is the abelianisation of the Weil group W_K.

We may now recast this statement in the following manner. Let $\mathcal{A}_1(K)$ denote the set of isomorphism classes of irreducible complex representations (ρ, V) of $K^\times = GL_1(K)$ with the following two properties:

- The stabiliser of v, $\text{stab}_{K^\times}(v)$, is open in $K^\times$ for every $v \in V$.
- The space of H-invariants in V,V^H, is finite dimensional for all open subgroups $H \subseteq K^\times$.

Now choose some $0 \neq v \in V$ and let $U := \text{stab}_{K^\times}(v)$. It then follows from these properties that V^U is finite dimensional. Since $K^\times$ is abelian, V^U must be stable under the group action and it follows that $V = V^U$ since the representation is irreducible. We will see in section 2 that every such representation (ρ, V) is necessarily one-dimensional, and so $\mathcal{A}_1(K)$ is precisely equal to the set of continuous homomorphisms $K^\times \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is endowed with the discrete topology.

Conversely, let $\mathcal{G}_1(K)$ denote the set of continuous homomorphisms $W_K \to \mathbb{C}^\times = GL_1(\mathbb{C})$, where $\mathbb{C}^\times$ is endowed with its usual topology. Recall that a homomorphism $W_K \to \mathbb{C}^\times$ is continuous if and only if its restriction to I_K is continuous. Since I_K is a subgroup of the totally disconnected space W_K, it too is also totally disconnected. Moreover by general theory I_K is compact, and so it follows that the image of I_K under such a homomorphism will be a compact and totally disconnected subgroup of $\mathbb{C}^\times$, which must necessarily be finite. We may therefore conclude that a homomorphism $W_K \to \mathbb{C}^\times$ is continuous for the usual topology of $\mathbb{C}^\times$ if and only if it is continous with respect to the discrete topology of $\mathbb{C}^\times$. Thus we may reformulate Theorem 1.1 in the following fashion:

Theorem 1.2. *(The Local Langlands Correspondence for GL_1) There exists a bijection between $\mathcal{A}_1(K)$, the set of isomorphism classes of irreducible complex representations of $GL_1(K)$, and $\mathcal{G}_1(K)$, the set of continuous homomorphisms $W_K \to GL_1(\mathbb{C})$.*

Local class field theory is but one simple example of the profound, far-reaching consequences the local Langlands correspondence has in mathematics. A thorough explanation and proof of the result in the case where $n = 2$ is given by Bushnell and Henniart in [5].

1.2. The Local Langlands Correspondence for GL_n.

We now give the statement of the local Langlands correspondence for GL_n in its full generality, following the statement given by Scholze in [37]. The objects involved are complicated and we shall devote the remainder of this essay to defining these.

Let $\mathcal{A}_n(K)$ denote the set of equivalence classes of irreducible admissible representations of $GL_n(K)$ (as defined in section 2), and let $\mathcal{G}_n(K)$ denote the set of equivalence classes of Frobenius semisimple n-dimensional complex Weil-Deligne representations of the Weil group W_K (as defined in 3). We denote by L and ϵ the corresponding L and ϵ-factors, which are defined in the section corresponding to the side of the bijection they lie on. Finally, by $(\ \ )^\vee$ we denote the corresponding contragredient (or dual) representation, which is also defined in both cases in the following sections.

Theorem 1.3. *(The local Langlands correspondence for GL_n) Fix a non-trivial additive character $\psi : K \to \mathbb{C}^\times$. Then there exists a unique collection of bijections:*

$$rec_n : \mathcal{A}_n(K) \to \mathcal{G}_n(K)$$

which satisfy the following five properties:

(a) *In degree 1 it is given by local class field theory. In other words, if $\rho \in \mathcal{A}_1(K)$ we have:*

$$rec_1(\rho) = \rho \circ Art_K^{-1}.$$

(b) *It is compatible with L and ϵ-factors, i.e for $\rho_1 \in \mathcal{A}_{n_1}(K)$ and $\rho_2 \in \mathcal{A}_{n_2}(K)$ we have:*

$$L(\rho_1 \times \rho_2, s) = L(rec_{n_1}(\rho_1) \otimes rec_{n_2}(\rho_2), s),$$
$$\epsilon(\rho_1 \times \rho_2, \psi, s) = \epsilon(rec_{n_1}(\rho_1) \otimes rec_{n_2}(\rho_2), \psi, s).$$

(c) *It is compatible with twisting by characters, i.e for $\rho \in \mathcal{A}_n(K)$ and $\chi \in \mathcal{A}_1(K)$ we have:*

$$rec_n(\rho\chi) = rec_n(\rho) \otimes rec_1(\chi).$$

(d) *It is compatible with central characters, i.e for $\rho \in \mathcal{A}_n(K)$ with central character ω_ρ we have:*

$$\det \circ rec_n(\rho) = rec_1(\omega_\rho).$$

(e) *It is compatible with contragredients, i.e for $\rho \in \mathcal{A}_n(K)$ we have:*

$$rec_n(\rho^\vee) = rec_n(\rho)^\vee.$$

Moreover, this collection is independent of our choice of the additive character ψ.

The crucial part of this theorem used in many applications is that certain properties of representations on one side of this bijection correspond to properties of representations on the other side, such as the aforementioned L and ϵ-factors. Some such properties are known and well-studied, but on the whole this is a broadly open problem.

Two examples of such invariants which are preserved under the bijections described above are the conductor $f(\rho)$, which we shall meet in Definition 2.32, and the depth $d(\rho)$, which was defined by Moy and Prasad in [32] and [34]. The depth is a number defined in terms of filtrations of parahoric subgroups on the GL_n side and in terms of ramification groups on the Galois side.

We will now proceed to define and explain all of these objects appearing in the statement of Theorem 1.3. We will also give some further directions in which this correspondence takes us, although without proof and instead giving reference.

2. Explanation of the GL_n Side

In this section we shall define everything involved in the GL_n side of Theorem 1.3. We begin by introducing the notion of an admissible representation and discuss some examples and properties of these. We then give a brief insight into the method of parabolic induction, which allows us to introduce supercuspidal representations. These are then discussed in the context of the Bernstein-Zelevinsky classification theorem, which may be used to prove Theorem 1.3 by reducing to the case of supercuspidal representations. Finally, we define the L and ϵ-factors on this side of the bijection and discuss how they are built and some of their important properties. The material treated in this exposition roughly corresponds to that of [5], [35] and [45].

Throughout this section, let V be a $\mathbb{C}$-vector space.

2.1. On Admissible Representations.

The theory addressed in this section concerns locally profinite groups. Recall that a topological group G is said to be *locally profinite* if it is Hausdorff and every open neighbourhood of the identity element contains a compact open subgroup. $GL_n(K)$ will be our prototypical example of such a group. Indeed, it is Hausdorff and also a topological group as it inherits the subspace topology induced from the product topology on $M_n(K)$. A basis of open neighbourhoods of the identity are given by the *congruence subgroups*:

$$K_0 := GL_n(\mathcal{O}_K), \qquad K_r := I_n + \pi_K^r M_n(\mathcal{O}_K) \text{ for } r \in \mathbb{N}.$$

In order to understand the statement of Theorem 1.3 we need only understand the cases where $G = GL_n(K)$ in the following. However, we shall avoid specialising to this case where possible in order to provide an exposition in full generality.

Definition 2.1. Let (ρ, V) be a representation of G. We say that ρ is *smooth* if $\text{stab}_G(v)$ is open in G for every $v \in V$.

Let (ρ, V) be a representation of G. We define:

$$V^\infty := \bigcup_K V^K,$$

where K ranges over all compact open subgroups of G. This is a G-stable subspace of V, and since G is locally profinite it follows that ρ is smooth if and only if $V = V^\infty$.

Definition 2.2. Let (ρ, V) be a smooth representation. We say that ρ is *admissible* if for every open subgroup $H \leq G$ the space:

$$V^H = \{v \in V : \rho(h)v = v \text{ for all } h \in H\}$$

of H-invariants in V is finite dimensional.

Example 2.3. We now give some examples of these types of representations.

(a) If (ρ, V) is any smooth representation with V finite dimensional, then ρ is automatically an admissible representation.
(b) The regular representation of $GL_n(K)$ on K^n is not smooth: indeed, for any non-zero $v \in V$ we have that $\text{stab}_G(v)$ is not open.
(c) The trivial representation $\mathbf{1}$ is admissible, whereas $\mathbf{1}^\infty$ is smooth but not admissible.
(d) In this example we prove that the "smooth regular representation" is an example of an admissible representation. Let p be a prime number and let $G = \mathbb{Z}_p$ be our locally profinite group. Then G acts naturally on the vector space:

$$V = \{f : \mathbb{Z}_p \to \mathbb{C} \mid f \text{ is locally constant}\}.$$

- **Smooth:** Let $f \in V$ be arbitrary. Since f is locally constant, for any $x \in G$ we may find a neighbourhood U_x of x such that $f|_{U_x}$ is constant. This yields a covering of G by open neighbourhoods of x of the form $U_{n_i} = x + p^{n_i}\mathbb{Z}_p$. Since G is compact, there is a finite subcovering $U_{n_1}, \ldots, U_{n_m}$. Let $n = max\{n_1, \ldots, n_m\}$. Then this means that f is fixed by $p^n\mathbb{Z}_p$, so the stabilizer of f is open. Since f was arbitrary, it follows that the representation is smooth.

- **Admissible:** For all $m \geq 0$ we have that $p^m\mathbb{Z}_p$ is a basis of open neighbourhoods of $0 \in G$. It follows that:

$$\begin{aligned} V^{p^m\mathbb{Z}_p} &= \{f : f \text{ is constant on } p^m\mathbb{Z}_p\text{-orbits}\} \\ &= \{f : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}\}, \end{aligned}$$

which is certainly finite dimensional. It follows that the representation is admissible.

The following result of Jacquet gives a characterisation of these types of representations.

Proposition 2.4. *Let (ρ, V) be an irreducible smooth complex representation of G. Then ρ is admissible.*

Proof. The proof is beyond the scope of this essay and is omitted. A detailed proof is presented in ([3], Theorem 3.25) for the case $G = GL_n(K)$, which makes use of a theorem of Harish-Chandra and the theory of quasi-cuspidal representations. □

We now consider Schur's Lemma and some of its consequences. The proof is fairly standard and may be found in ([5], Chapter 2).

Lemma 2.5. *Let (ρ, V) be an irreducible admissible representation of G. Then $End_G(V) = \mathbb{C}$.*

Let (ρ, V) be an irreducible admissible representation of G. Then as a consequence of Schur's lemma, the centre $Z(G)$ acts on V by a character $\omega_\rho : Z(G) \to \mathbb{C}^\times$, i.e $\rho(z)v = \omega_\rho(z)v$ for all $v \in V$ and $z \in Z(G)$. We call ω_ρ the *central character* of ρ.

Corollary 2.6. *Let (ρ, V) be an irreducible admissible representation of $GL_n(K)$. Then V is either 1-dimensional or infinite-dimensional. In particular, if V is 1-dimensional then:*

$$\rho = \chi \circ \det$$

for some quasi-character $\chi : K^\times \to \mathbb{C}^\times$.

Proof. Let (ρ, V) be an irreducible admissible finite dimensional representation of $GL_n(K)$. Since ρ is smooth, we have that $\ker(\rho) \trianglelefteq GL_n(K)$ is an open subgroup and so it contains $I_n + \pi_K^m M_n(\mathcal{O}_K)$ for some $m \in \mathbb{N}$. In particular, it therefore contains all of the unipotent upper triangular matrices, whose diagonal entries are all 1 with the rest in $\pi_K^m\mathcal{O}_K$. Moreover, by normality we can conjugate these matrices to see that $SL_n(K) \subseteq \ker(\rho)$, and so ρ has to factor through $GL_n(K)/SL_n(K) \cong K^\times$. Since the determinant map is surjective on $GL_n(K)$, the result follows. Finally, Schur's lemma then allows us to conclude that V is 1-dimensional. □

Let (ρ, V) be a smooth representation of G. We now wish to define an associated smooth representation to the dual space V^* of V. To do so, let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ be given by $\langle v^*, v \rangle = v^*(v)$. Then V^* admits a natural representation ρ^* which is defined by:

$$\langle \rho^*(g)v^*, v \rangle = \langle v^*, \rho(g^{-1})v \rangle, \quad \forall\, g \in G, v^* \in V^*, v \in V.$$

This is not necessarily smooth, so to remedy this we define our way out by setting $V^\vee := (V^*)^\infty$. Since this is a G-stable subspace of V^* we therefore induce a smooth representation $(\rho^\vee, V^\vee)$ of G, which we call the *contragredient* of (ρ, V).

The following theorem of Gel'fand and Kazhdan ([3], Theorem 7.3) gives a description of the contragredient in the case where $G = GL_n(K)$.

Theorem 2.7. *Define the matrix $s_n \in GL_n(K)$ and an automorphism $g \mapsto {}^s g$ of $GL_n(K)$ by:*

$$(s_n)_{ij} := (-1)^j \cdot \delta_{i,n+1-j} \quad \text{and} \quad {}^s g := s_n (g^T)^{-1} s_n^{-1}.$$

If ρ is an irreducible admissible representation then $\rho^\vee$ is isomorphic to the representation ρ^s given by $g \mapsto \rho({}^s g^{-1})$.

2.2. **Parabolic Induction and Supercuspidals.** We now proceed to give an overview of the method of parabolic induction and use this in order to define the notion of a supercuspidal representation, which may be thought of as the building blocks of irreducible admissible representations. In this section we follow the exposition given by Kudla [26], which summarises the techniques developed by Bernstein and Zelevinsky in [2], [3] and [47]. We now forego the locally profinite group generalities and focus on the case where $G = GL_n(K)$.

Definition 2.8. Let $\underline{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ be an ordered partition of n. The *standard parabolic subgroup* $P_{\underline{n}}$ of $GL_n(K)$ is the subgroup consisting of matrices of the form:

$$\begin{pmatrix} M_1 & & & \\ & M_2 & * & \\ & & \cdots & \\ & 0 & \cdots & \\ & & & M_r \end{pmatrix},$$

where each $M_i \in GL_{n_i}(K)$. The *unipotent radical* $U_{\underline{n}}$ of $P_{\underline{n}}$ is the normal subgroup consisting of matrices of the form:

$$\begin{pmatrix} I_{n_1} & & & \\ & I_{n_2} & * & \\ & & \cdots & \\ & 0 & \cdots & \\ & & & I_{n_r} \end{pmatrix}.$$

Then $P_{\underline{n}} = L_{\underline{n}} \ltimes U_{\underline{n}}$, where $L_{\underline{n}}$ is the *Levi subgroup* of $GL_n(K)$, consisting of matrices of the form:

$$\begin{pmatrix} M_1 & & & \\ & M_2 & 0 & \\ & & \cdots & \\ & 0 & \cdots & \\ & & & M_r \end{pmatrix},$$

where each $M_i \in GL_{n_i}(K)$ so that:

$$L_{\underline{n}} \cong \prod_{i=1}^{r} GL_{n_i}(K),$$

in the natural way. This is called the *Levi decomposition* of $P_{\underline{n}}$.

For each $i \in \{1, \ldots, r\}$ let (ρ_i, V_i) be an admissible representation of $GL_{n_i}(K)$. The tensor product of admissible representations is again an admissible representation ([2], 1.6), and so it follows that $\rho := \rho_1 \otimes \cdots \otimes \rho_r$ is an admissible representation of $L_{\underline{n}}$ on $W := V_1 \otimes \cdots \otimes V_r$.

Since $L_{\underline{n}}$ is the quotient of $P_{\underline{n}}$ by $U_{\underline{n}}$, we may proceed to inflate ρ to get a representation of $P_{\underline{n}}$. Formally, if we let $\pi : P_{\underline{n}} \to P_{\underline{n}}/U_{\underline{n}} = L_{\underline{n}}$ denote the canonical projection then we write:

$$\mathrm{Inf}_{L_{\underline{n}}}^{P_{\underline{n}}}(\rho) := \rho \circ \pi : P_{\underline{n}} \to GL_n(W).$$

For notational reasons we shall denote this representation by ρ as well. This procedure is often referred to as 'extending the representation trivially across $U_{\underline{n}}$' in the literature.

Let $\Delta_{GL_n(K)}$ denote the modular character of $GL_n(K)$. We define:

$$\Delta := \Delta_{GL_n(K)}|_{U_{\underline{n}}} \cdot \Delta_{U_{\underline{n}}}^{-1}.$$

Using this we may now associate the *normalised induced representation* $\mathrm{Ind}_{P_{\underline{n}}}^{GL_n(K)}(\rho)$ ([45]) which gives a representation of $GL_n(K)$ with underlying representation space as follows:

$$V = \left\{ f \colon GL_n(K) \to W \;\middle|\; \begin{array}{l} (1)\;\; f(pg) = \Delta(p)^{1/2}\rho(p)f(g), \forall p \in U_{\underline{n}}, g \in GL_n(K) \\ (2)\;\; f(gu) = f(g) \text{ for all } u \in U_f, \text{ an open set in } GL_n(K) \end{array} \right\}$$

The term 'normalised' refers to the appearance of the $\Delta^{1/2}$ factor in (1). This term is used in practise to simplify certain formulae, with the drawback of significantly complicating some others such as Frobenius reciprocity. Possible references for further examples and for more on parabolic induction are [4] or [44].

We summarise the method in the following definition.

Definition 2.9. Let $(n_1, \ldots, n_r)$ be an ordered partition of n. For each $i \in \{1, \ldots, r\}$ suppose that (ρ_i, V_i) is an admissible representation of $GL_{n_i}(K)$. We define the *product representation of* $\rho_1, \ldots, \rho_r$ as:

$$\rho_1 \times \cdots \times \rho_r := \mathrm{Ind}_{P_{\underline{n}}}^{GL_n(K)} \left(\mathrm{Inf}_{L_{\underline{n}}}^{P_{\underline{n}}} (\rho_1 \otimes \cdots \otimes \rho_r) \right),$$

and we say that this representation is *parabolically induced* from $L_{\underline{n}}$ to $GL_n(K)$.

We take as a fact ([3], Lemma 2.26) that $\rho_1 \times \cdots \times \rho_r$ is again an admissible representation. The proof of this relies on the compactness of $U_{\underline{n}}$ and various results concerning induction.

We may now define supercuspidal representations, following [45].

Definition 2.10. Let (ρ, V) be an irreducible admissible representation of $GL_n(K)$. We say that ρ is a *supercuspidal representation* if there is no proper partition $\underline{n} = (n_1, \ldots, n_r)$ such that ρ is a subquotient of a product representation $\rho_1 \times \cdots \times \rho_r$, where each ρ_i is an admissible representation of $GL_{n_i}(K)$.

We note in passing that the irreducible admissible representations which can not be obtained in this way are known as *non-cuspidal representations*, or *principal series*. As an example, the principal series for $GL_2(K)$ are rather straightforward to describe - this classification is presented in ([5], Section 9). The theory of principal series are not required in what follows and so they will not be mentioned again. Possible references to explore these types of representations further are [4] and [22].

Definition 2.11. Let (ρ, V) be a smooth representation of $GL_n(K)$, and let $v \in V, v^\vee \in V^\vee$. The *matrix coefficient* associated to the pair $(v, v^\vee)$ is the map:

$$\begin{aligned} c_{\rho,v,v^\vee} : GL_n(K) &\to \mathbb{C} \\ g &\mapsto \langle v^\vee, \rho(g)v \rangle. \end{aligned}$$

So far our definition of a supercuspidal representation is highly abstract and not particularly useful. The next theorem allows us to gain some technical control on the property of being supercuspidal by relating this to a property regarding the matrix coefficients, in the following sense.

Theorem 2.12. *Let ρ be a irreducible admissible representation of $GL_n(K)$. Then the following statements are equivalent:*

- *(a) ρ is a supercuspidal representation.*
- *(b) $\rho^\vee$ is a supercuspidal representation.*
- *(c) For all $v \in V, v^\vee \in V^\vee$ the matrix coefficient $c_{\rho,v,v^\vee}$ has compact support modulo the centre.*
- *(d) For all quasi-characters $\chi : GL_n(K) \to K^\times$, the twisted representation $\rho\chi$ is a supercuspidal representation (as in Definition 2.13).*

Moreover, if $\rho^\vee$ is also irreducible then (a)-(d) are equivalent to:

- *(e) There exists a pair $v \in V, v \in V^\vee$ such that $c_{\rho,v,v^\vee}$ has compact support modulo the centre.*

Proof. The proof is beyond the scope of this essay and is omitted. The equivalence of (a) and (c) is a result due to Harish-Chandra ([3], Theorem 3.21). The equivalence of the other results follow from various results presented by Bernstein and Zelevinsky in [3], and are exposited further in [45]. □

2.3. **The Bernstein-Zelevinsky Classification.** We are now in a position to introduce the Bernstein-Zelevinsky classification theorem for $GL_n(K)$. This is a powerful result which allows us to relate irreducible admissible representations to supercuspidal representations. All of the results here appear in [2] and [47], which we follow closely. First we make a preliminary definition.

Definition 2.13. Let (ρ, V) be a smooth representation of G and let χ be a quasi-character of G. The *twisted representation* $\rho\chi$ is defined by $g \mapsto \rho(g)\chi(g)$.

Let (ρ, V) be an admissible representation of $GL_n(K)$. For any $s \in \mathbb{C}$, we consider the twist of ρ with the character $|\cdot|^s : GL_n(K) \to \mathbb{C}^\times$, defined by $\rho(s)(g) = |\det(g)|^s \rho(g)$. It follows that if ρ is a supercuspidal representation of $GL_n(K)$ then so is $\rho(s)$.

We may then define a partial order on the set of isomorphism classes of supercuspidal representations of $GL_n(K)$ by writing $\rho \leq \rho' \Leftrightarrow \rho' = \rho(s)$ for some non-negative integer s. We shall denote this poset by $\mathcal{A}_n^{\mathrm{sc}}$.

Definition 2.14. A *Bernstein-Zelevinsky Segment* starting from ρ of length m and degree mn is a finite ordered list of m supercuspidal representations of $GL_n(K)$:

$$\Delta(\rho, m) = [\rho, \rho(1), \ldots, \rho(m-1)].$$

Definition 2.15. Let $\Delta_1 = \Delta(\rho_1, n_1) \in \mathcal{A}_{n_1}^{\mathrm{sc}}$ and $\Delta_2 = \Delta(\rho_2, n_2) \in \mathcal{A}_{n_2}^{\mathrm{sc}}$ be two Bernstein-Zelevinsky segments. We say that Δ_1 and Δ_2 are *linked* if $\Delta_1 \subsetneq \Delta_2, \Delta_2 \subsetneq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is also a Bernstein-Zelevinsky segment. We also say that Δ_1 *precedes* Δ_2 if they are linked and $\rho_1 < \rho_2$.

A trivial yet important observation we can make is that in order for $\Delta_1 \in \mathcal{A}_{n_1}^{\mathrm{sc}}$ and $\Delta_2 \in \mathcal{A}_{n_2}^{\mathrm{sc}}$ to be linked (and hence to precede each other) we must necessarily have that $n_1 = n_2$.

Definition 2.16. Let $\Delta = \Delta(\rho, m)$ be a Bernstein-Zelevinsky segment of length m and degree mn. We may then define an admissible representation of $GL_{mn}(K)$ by:

$$\rho(\Delta) := \rho \times \cdots \times \rho(m-1).$$

Now we are able to state the main results of the classification theorem found in [2] and [47].

Theorem 2.17. *(Bernstein-Zelevinsky classification):*

(a) Let Δ be a Bernstein Zelevinsky segment of length m. Then the representation $\rho(\Delta)$ has length 2^{m-1}. Moreover, it has a unique irreducible quotient representation $Q(\Delta)$ and a unique irreducible subrepresentation $Z(\Delta)$.

(b) Let $\Delta_1 \subseteq \mathcal{A}_{n_1}^{sc}, \ldots, \Delta_r \subseteq \mathcal{A}_{n_r}^{sc}$ be Bernstein-Zelevinsky segments such that for every $i < j, \Delta_i$ does not precede Δ_j. Then the representation $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ admits a unique irreducible quotient representation $Q(\Delta_1, \ldots, \Delta_r)$. Moreover, the representation $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ admits a unique irreducible subrepresentation $Z(\Delta_1, \ldots, \Delta_r)$.

(c) Let ρ be an irreducible admissible representation of $GL_n(K)$. Then it is isomorphic to a representation of the form $Q(\Delta_1, \ldots, \Delta_r)$ for a unique collection of Bernstein-Zelevinsky intervals $\Delta_1, \ldots, \Delta_r$ such that Δ_i does not precede Δ_j for any $i < j$.

(d) Under the hypothesis of (b), the representation $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ is irreducible if and only if no two of the intervals Δ_i and Δ_j are linked.

In the literature $Q(\Delta_1, \ldots, \Delta_r)$ is often called the *Langlands quotient*.

It is a fact that given any r-tuple of Bernstein-Zelevinsky segments $\Delta_i = [\rho_i, \ldots, \rho_i(m_i - 1)]$, we are always able to uniquely order them so that Δ_i does not precede Δ_j for $i < j$. It then follows from (b) and (c) of Theorem 2.17 that the set of isomorphism classes of irreducible admissible representations of $GL_n(K)$ is in bijection with the set of unordered r-tuples $(\Delta_1(\rho_1, m_1) \ldots, \Delta_r(\rho_r, m_r))$ with each $\rho_i \in \mathcal{A}_{n_i}^{\mathrm{sc}}$ and $n = \sum_{i=1}^{r} m_i n_i$.

Definition 2.18. Let $\rho \cong Q(\Delta_1, \dots, \Delta_r)$ be an irreducible admissible representation of $GL_n(K)$. We define the *supercuspidal support* of ρ to be the unordered collection of supercuspidal representations corresponding to ρ via the bijection above. Namely:

$$\mathrm{Supp}(\rho) := \{\rho_i(j)\},$$

for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, m_i - 1\}$.

An important example of such a representation is the following.

Example 2.19. Consider the character $|\cdot|^{\frac{1-n}{2}} : GL_n(K) \to \mathbb{C}^\times$ and define the Bernstein-Zelevinsky segment:

$$\Delta(|\cdot|^{\frac{1-n}{2}}, n) = \left[|\cdot|^{\frac{1-n}{2}}, \cdots, |\cdot|^{\frac{n-1}{2}} \right].$$

We call the corresponding quotient representation $Q(\Delta)$ the *Steinberg representation of* $GL_n(K)$, denoted $St(n)$. It corresponds to the twist $|\cdot|^{\frac{1-n}{2}} Sp(n)$ (see Example 3.8) under rec_n ([45], 4.3.4).

Remark 2.20. The most important application of Theorem 2.17 for us is noting that it allows for a simplification of the proof of Theorem 1.3 by reducing the problem to considering the supercuspidal representations. Indeed, by this classification Henniart ([19]) proved that it suffices to prove that there is a unique collection of bijections:

$$\widetilde{\mathrm{rec}_n} : \mathcal{A}_n^{\mathrm{sc}} \to \mathcal{G}_n^{\mathrm{irr}},$$

satisfying the five properties laid out in Theorem 1.3, where $\mathcal{G}_n^{\mathrm{irr}}$ denotes the set of isomorphism classes of irreducible Weil-Deligne representations of the Weil group W_K.

The map is defined as follows. Let $\rho \cong Q(\Delta_1, \dots, \Delta_r)$ be an irreducible admissible representation of $\mathrm{GL}_n(K)$ with corresponding Bernstein-Zelevinsky segments $\Delta_i = \Delta_i(\rho_i, m_i)$ with each $\rho_i \in \mathcal{A}_{n_i}^{\mathrm{sc}}$ and $n = \sum_{i=1}^r n_i m_i$. Then we define:

$$\widetilde{\mathrm{rec}_n}(\rho) = \bigoplus_{i=1}^r \left(\mathrm{rec}_{n_i}(\rho_i) \otimes \mathrm{Sp}(m_i) \right),$$

where Sp is as defined in Example 3.8. One may verify that the five properties laid out in Theorem 1.3 are then satisfied, which follows from analysis of the bad reduction of certain Shimura varieties. We refer the reader to [17] for full details and justification.

2.4. $L + \epsilon$ **factors.** To conclude this section we shall define the L and ϵ-factors on this side of the bijection, following the theory developed in [23] and presented in [18]. Our exposition roughly follows the order given in [26] and [45].

Many of the results and definitions in this section follow from those first developed in Tate's thesis ([43]). The main object of study in his thesis are L-functions which are ubiquitous in the study of number theory. Tate gives an explication of the functional equation of L-series in terms of local functional equations, which are discussed in [42]. Such important results will motivate our subsequent discussions and are well exposited with examples by Kudla in [27].

Let $\rho \in \mathcal{A}_n$ and $\rho' \in \mathcal{A}_{n'}$ be irreducible admissible representations of $GL_n(K)$ and $GL_{n'}(K)$, respectively, and let $U_n(K) \leq GL_n(K)$ denote the unipotent upper triangular matrices. Choose a non-trivial additive character $\psi : K \to \mathbb{C}$. We may then define an associated one-dimensional representation $\theta_\psi : U_n(K) \to \mathbb{C}$ by:

$$\theta_\psi \left(\begin{pmatrix} 1 & & u_{i,j} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

Definition 2.21. Let ρ be an irreducible admissible representation of $GL_n(K)$. We say that ρ is a *generic representation* if:

$$\mathrm{Hom}_{U_n(K)}(\rho|_{U_n(K)}, \theta_\psi) \neq 0.$$

Generic representations are fundamental in the theory of L-functions - for further details and exploration of this topic, see [21] or [40].

Following Theorem 2.17, we may write $\rho \cong Q(\Delta_1, \ldots, \Delta_r)$. Bernstein and Zelevinsky prove ([47], Theorem 9.7) that in that case, ρ is a generic representation if and only if none of the Δ_i are linked. It follows from Theorem 2.17 (d) that if ρ is generic then $\rho \cong Q(\Delta_1) \times \cdots \times Q(\Delta_r)$. As a consequence of this, every supercuspidal representation is generic. Finally, the property of being a generic representation is in fact independent of our choice of ψ ([45], Proposition 2.3.2).

Now we may introduce Whittaker models following ([22], Chapter 1).

Definition 2.22. Let (ρ, V) be a generic representation. A *Whittaker functional for* ρ is a map $\lambda : V \to \mathbb{C}$ such that for all $u \in U_n(K), v \in V$ we have:

$$\lambda(\rho(u)v) = \theta_\psi(u)\lambda(v).$$

We shall take the existence of Whittaker functionals for granted. Fixing a Whittaker functional λ for ρ, we may define the *Whittaker model for* ρ *with respect to* ψ by:

$$\mathcal{W}(\rho, \psi) := \{W_v : GL_n(K) \to \mathbb{C} \mid W_v(g) = \lambda(\rho(g)v)\},$$

together with a $GL_n(K)$-action given by $g \cdot W_v = W_{gv}$. Up to isomorphism, this is in fact independent of our choice of ψ ([22]). We then have the following result of Shalika as presented in [41], known as the *multiplicity one theorem*, which states that:

Theorem 2.23. *The dimension of the space of Whittaker functionals is at most one.*

As an immediate consequence of this result, it follows that if ρ admits a Whittaker model then it is necessarily unique. With this theory in hand we are ready to introduce the L and ϵ-factors, following results presented in [23] and [24].

From now on assume that ρ and ρ' are generic and that ψ is unitary. To begin with we shall consider the case where $n = n'$. We define the *Schwartz space on* K^n by:

$$\mathcal{S}(K^n) := \{f : K^n \to \mathbb{C} \mid f \text{ is locally constant with compact support}\}.$$

Let dg be a $GL_n(K)$-invariant measure on $U_n(K)\backslash GL_n(K)$. Then we define:

$$Z(W, W', f, s) := \int_{U_n(K)\backslash GL_n(K)} W(g)W'(g)f((\underbrace{0, \ldots, 0}_{n-1 \text{ 0's}}, 1)g)|\det(g)|^s \, dg,$$

for all $W \in \mathcal{W}(\rho, \psi), W' \in \mathcal{W}(\rho', \overline{\psi})$ and $f \in \mathcal{S}(K^n)$. This is then a rational function of q^{-s} which converges for $\mathrm{Re}(s) \gg 0$. Furthermore define:

$$I := \{Z(W, W', f, s) \mid W \in \mathcal{W}(\rho, \psi), W' \in \mathcal{W}(\rho', \overline{\psi}) \text{ and } f \in \mathcal{S}(K^n)\}.$$

Consider the fractional ideal generated by I in $\mathbb{C}[q^{-s}, q^s]$. It is a fact that there exists a unique polynomial $P \in \mathbb{C}[x]$ with $P(0) = 1$ such that $P(q^{-s})^{-1}$ uniquely generates this ideal.

Definition 2.24. In the case where $n = n'$ we define the *L-factor associated to ρ and ρ'*, denoted $L(\rho \times \rho', s)$, to be the unique generator of the fractional ideal generated by I in $\mathbb{C}[q^{-s}, q^s]$.

Now we may proceed to define the ϵ-factors. Let $w_n \in GL_n(K)$ be the permutation matrix such that $w_n(e_i) = e_{n+1-i}$ for all $i \in \{1, \ldots, n\}$, otherwise known as the longest element of the corresponding Weyl group. By Theorem 2.7, for each $W \in W(\rho, \psi)$ we may define $\widetilde{W} \in W(\rho^\vee, \overline{\psi})$ by $\widetilde{W}(g) = W(w_n \, {}^t g^{-1})$ and similarly $\widetilde{W'} \in W(\rho'^\vee, \psi)$ is defined by $\widetilde{W'}(g) = W'(w_n \, {}^t g^{-1})$. If $f \in \mathcal{S}(K^n)$ then we define the *Fourier transform of* f *with respect to* ψ, denoted $\hat{f}$, by:

$$\hat{f}(x) = \int_{K^n} f(y)\psi({}^t(yx))dy,$$

for $x \in K^n$.

Definition 2.25. In the case where $n = n'$ we define the *ϵ-factor associated to ρ and ρ'*, denoted $\epsilon(\rho \times \rho', \psi, s)$, to be the monomial in q^{-s} defined by the 'Tate functional equation':

$$\frac{Z(\widetilde{W}, \widetilde{W'}, \hat{f}, 1-s)}{L(\rho^\vee \times \rho'^\vee, 1-s)} = \omega_{\rho'}(-I_n)^n \epsilon(\rho \times \rho', \psi, s) \frac{Z(W, W', f, s)}{L(\rho \times \rho', s)},$$

where $\omega_{\rho'}$ denotes the central character of (ρ', V').

Now consider the case where $n' < n$. Once again let $W \in \mathcal{W}(\rho, \psi), W' \in \mathcal{W}(\rho', \overline{\psi})$ and $j \in \{0, 1, \ldots, n-n'-1\}$. Moreover, let dg be a $GL_{n'}(K)$-invariant measure on $U_{n'}(K)/GL_{n'}(K)$ and let dx be a Haar measure on $M_{j\times n'}(K)$. We define:

$$Z(W, W', j, s) := \int_{U_{n'}(K)\backslash GL_{n'}(K)} \int_{M_{j\times n'}(K)} W\left(\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-n'-j} \end{pmatrix}\right) W'(g) |\det(g)|^{s-\frac{n-n'}{2}}\, dx\, dg.$$

Again, this is a rational function of q^{-s} which converges for $\mathrm{Re}(s) \gg 0$. We similarly define:

$$I' := \{Z(W, W', j, s) \mid W \in \mathcal{W}(\rho, \psi), W' \in \mathcal{W}(\rho', \overline{\psi}) \text{ and } j \in \{0, 1, \ldots, n-n'-1\}\}.$$

Similarly to the previous discussion, I' generates a fractional ideal in $\mathbb{C}[q^{-s}, q^s]$. There exists a unique polynomial $P \in \mathbb{C}[x]$ with $P(0) = 1$ such that $P(q^{-s})^{-1}$ uniquely generates this ideal.

Definition 2.26. In the case where $n' < n$ we define the *L-factor associated to ρ and ρ'*, denoted $L(\rho \times \rho', s)$, to be the unique generator of the fractional ideal generated by I' in $\mathbb{C}[q^{-s}, q^s]$.

Now define the matrix:

$$w_{n,n'} := \begin{pmatrix} I_{n'} & 0 \\ 0 & w_{n-n'} \end{pmatrix} \in GL_n(K).$$

We may then use this to define the ϵ-factors in this case in a similar fashion.

Definition 2.27. In the case where $n' < n$ we define the *ϵ-factor associated to ρ and ρ'*, denoted $\epsilon(\rho \times \rho', \psi, s)$, to be monomial in q^{-s} defined by the 'Tate functional equation':

$$\frac{Z(w_{n,n'}\widetilde{W}, \widetilde{W'}, n-n'-1-j, 1-s)}{L(\rho^\vee \times \rho'^\vee, 1-s)} = \omega_{\pi'}(-1)^{n-1} \epsilon(\rho \times \rho', \psi, s) \frac{Z(W, W', j, s)}{L(\rho \times \rho', s)},$$

Finally in the case where $n < n'$ we just swap the places of ρ and ρ' and use the definitions given above for the cases where $n' < n$.

Definition 2.28. In the case where $n < n'$ we define:

$$L(\rho \times \rho', s) := L(\rho' \times \rho, s)$$

and:

$$\epsilon(\rho \times \rho', \psi, s) := \epsilon(\rho' \times \rho, \psi, s).$$

Note that the omission of ψ in the notation of L-factors is incredibly suggestive. Indeed, the L-factors are independent of our choice of ψ ([45], 2.5.1) in each of the three cases mentioned above.

The definitions of L and ϵ-factors given above may appear to have been pulled out of a hat but are in fact motivated by work done by Tate in [43]. In particular the functional equations which appear in the definition of the L-factors arise from his considerations of local and global L-factors and zeta integrals. Possible references for further reading in this area are [9] or [42].

We may make use of the Bernstein-Zelevinsky classification to inductively define the L and ϵ-factors for general irreducible admissible representations using the following two results presented in Kudla ([26], Section 3.1).

Proposition 2.29.

(a) Let $\rho \cong Q(\Delta_1, \ldots, \Delta_r)$ and let ρ' be an irreducible admissible representation. Then:

$$L(\rho \times \rho', s) = \prod_{i=1}^{r} L(Q(\Delta_i) \times \rho', s),$$
$$\epsilon(\rho \times \rho', \psi, s) = \prod_{i=1}^{r} \epsilon(Q(\Delta_i) \times \rho', \psi, s).$$

(b) Suppose that $\rho \cong Q(\Delta)$ for $\Delta = \Delta(\rho, r)$ and $\rho' \cong Q(\Delta')$ for $\Delta' = \Delta'(\rho', r')$ with $r \le r'$. Then:

$$L(\rho \times \rho', s) = \prod_{i=1}^{r} L(\rho \times \rho', s + r + r' - 1)$$
$$\epsilon(\rho \times \rho', \psi, s) = \prod_{i=1}^{r} \left(\left(\prod_{j=0}^{r+r'-2i} \epsilon(\rho \times \rho', \psi, s + i + j - 1) \right) \right.$$
$$\left. \times \left(\prod_{j=0}^{r+r'-2i-1} \frac{L(\rho^\vee \times \rho'^\vee, 1 - s - i - j)}{L(\rho \times \rho', s + i + j - 1)} \right) \right).$$

We may also define L and ϵ-factors associated to a single representation as follows.

Definition 2.30. Let $\underline{1} : K^\times \to \mathbb{C}^\times$ be given by $\underline{1}(k) = 1$ for all $k \in K^\times$. For any irreducible admissible representation ρ of $GL_n(K)$, we define:

$$L(\rho, s) := L(\rho \times \underline{1}, s),$$
$$\epsilon(\rho, \psi, s) := \epsilon(\rho \times \underline{1}, \psi, s).$$

We also have the following simple description of L-factors for supercuspidal representations.

Proposition 2.31. *If ρ, ρ' are supercuspidal representations then:*

$$L(\rho \times \rho', s) = \prod_{\chi} L(\chi, s),$$

where χ runs over characters of $K^\times$ such that $(\rho')^\vee \chi = \rho$.

We should certainly not expect to find a simple formula like this for $\epsilon(\rho \times \rho', \psi, s)$. Indeed, by part (b) of Theorem 1.3 this corresponds to $\epsilon(\mathrm{rec}_{n_1}(\rho) \otimes \mathrm{rec}_{n_2}(\rho'), \psi, s)$, which is dependent on how $\rho \otimes \rho'$ splits into irreducibles, which could be complicated. However, the L-factor corresponds to $L(\mathrm{rec}_{n_1}(\rho) \otimes \mathrm{rec}_{n_2}(\rho'))$ by (b) which should only depend on the one-dimensional irreducible components of $\rho \otimes \rho'$, which agrees with Proposition 2.31.

We conclude by defining the conductor, which is an example of a property preserved under Theorem 1.3. Notably, Henniart uses this fact in his 'numerical local Langlands theorem' ([19]).

Definition 2.32. Let (ρ, V) be an irreducible admissible representation of $GL_n(K)$. For any non-negative integer t we define:

$$K_n(t) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathcal{O}_K) \mid c \in M_{1 \times (n-1)}(\pi_K^t \mathcal{O}_K), d \equiv I_{n-1} (\mathrm{mod}\ \pi_K^t M_n(\mathcal{O}_K)) \right\}.$$

The smallest t such that $V^{K_n(t)} \ne \{0\}$ is called the *conductor of* ρ, denoted $f(\rho)$.

The following theorem is due to work by Jacquet, Shalika and Piatetski-Shapiro ([24]) and allows us to relate the conductor to ϵ-factors.

Theorem 2.33. *Let ρ be generic and let $n(\psi)$ be the exponent of ψ (see Definition 3.27). Then:*

$$\epsilon(\rho, \psi, s) = \epsilon(\rho, \psi, 0) q^{-s(f(\pi) + n \cdot n(\psi))}.$$

3. Explanation of the Galois Side

In this section we shall define the objects involved in the Galois side of Theorem 1.3. We begin by recalling some basic number theoretic definitions and notions which we will need for the rest of the section. We then introduce Weil-Deligne representations and some basic results concerning these, as well as some examples. We then move on to discuss their relationship with ℓ-adic representations by proving results such as Grothendieck's ℓ-adic monodromy theorem. To conclude we give a brief introduction to the theory of L and ϵ-factors, which are special properties preserved by the correspondence. Henceforth we shall assume that the reader is familiar with the basic theory of profinite groups, a good reference for which is [46].

3.1. Prolegomena.

We begin by recalling some essential preliminaries which we shall refer to throughout this section. Possible references for further details are [20].

Definition 3.1. The *arithmetic Frobenius* $\sigma_q \in G_k$ is the map given by:

$$\sigma_q(x) = x^q,$$

The *geometric Frobenius* is the map given by:

$$\Phi_q := \sigma_q^{-1} \in G_k.$$

Recall that there is an isomorphism $G_k \cong \hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, where we stipulate that Φ_q is identified with 1. Moreover, recall that there is a surjective map given by restriction:

$$\mathrm{res} : G_K \twoheadrightarrow G_k.$$

This then allows us to give the following definitions.

Definition 3.2. The *inertia subgroup of* K is:

$$I_K := \ker(\mathrm{res}).$$

It is compact and has a maximal pro-p subgroup called the *wild inertia group*, denoted P_K.

We may then introduce the Weil group as follows.

Definition 3.3. The *Weil group* of K, denoted W_K, is a topological group defined in the following manner:

- As a group:

$$W_K = \mathrm{res}^{-1}(\langle \Phi \rangle)$$

- The topology is defined so that I_K is an open subgroup equipped with the profinite topology.

Then I_K and W_K fit inside of a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_K & \hookrightarrow & W_K & \twoheadrightarrow & \langle \Phi_q \rangle & \longrightarrow & 0 \\ & & \| \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_K & \twoheadrightarrow & G_K & \twoheadrightarrow & G_k & \longrightarrow & 0 \end{array}$$

W_K can therefore be generated by I_K and a lift of the geometric Frobenius $\Phi \in W_K$.

We may now make use of Theorem 1.1 to define a map $|\cdot|$ on W_K.

Definition 3.4. Define the *norm character* $|\cdot| : W_K \to K^\times$ by:

$$\begin{aligned} |\cdot| : W_K &\to \mathbb{R} \\ x &\mapsto |\mathrm{Art}_K^{-1}(x)|_K, \end{aligned}$$

where $|\cdot|_K$ is the unique absolute value on K extending the one on $\mathbb{Q}_p$. In particular if $\Phi \in W_K$ is a lift of the geometric Frobenius then we have $|\Phi| = q^{-1}$, and $|x| = 1$ for all $x \in I_K$.

3.2. **Weil-Deligne Representations.** With this basic theory in hand we may now define Weil-Deligne representations. We shall closely follow [42] throughout this section.

Definition 3.5. A *Weil-Deligne representation of* W_K is a triple (ρ, V, N) where (ρ, V) is a continuous (with respect to the discrete topology on $GL(V)$) finite-dimensional complex representation of W_K together with an endomorphism $N \in \mathrm{End}_{\mathbb{C}}(V)$ such that for all $w \in W_K$:

$$\rho(w)N = |w|N\rho(w).$$

A Weil-Deligne representation (ρ, V, N) is said to be *Frobenius semisimple* if (ρ, V) is semisimple.

Remark 3.6. Given a Weil-Deligne representation (ρ, V, N), it follows from this definition that the kernel of N is stable under W_K. Hence if ρ is irreducible we must necessarily have that $N = 0_{\mathrm{End}_{\mathbb{C}}(V)}$.

An immediate consequence of this definition is that N is automatically nilpotent.

Proposition 3.7. *Let (ρ, V, N) be a Weil-Deligne representation of W_K. Then N is nilpotent.*

Proof. Fix a lift of the geometric Frobenius $\Phi \in W_K$. Then we compute that:

$$\begin{aligned} \rho(\Phi)N &= |\Phi|N\rho(\Phi) \\ &= q^{-1}N\rho(\Phi), \end{aligned}$$

so $N\rho(\Phi) = q\rho(\Phi)N$. Let λ be an eigenvalue of N with respect to some vector $v \in V$. Then:

$$\begin{aligned} & N\rho(\Phi)v = q\rho(\Phi)Nv \\ \implies & N\rho(\Phi)v = q\rho(\Phi)\lambda v \\ \implies & N\rho(\Phi)v = q\lambda\rho(\Phi)v \end{aligned}$$

It follows that $\rho(\Phi)v$ is an eigenvector of N with corresponding eigenvalue $q\lambda$. A straightforward induction on n shows that $\rho^n(\Phi)v$ is an eigenvector of N with eigenvalue $q^n\lambda$ for every $n \in \mathbb{N}$. Observe that since V is finite dimensional, we cannot have $\lambda \neq 0$. Indeed, in that case we would obtain infinitely many eigenvectors of N. However, eigenvectors relative to different eigenvalues are linearly independent to each other, so this is impossible. We must therefore have that every eigenvalue of N is zero, and so N is nilpotent as claimed. □

We now give a standard construction and example of a Weil-Deligne representation.

Example 3.8. Let (ρ, V) be a continuous finite-dimensional representation of W_K. In that case we trivially have that $(\rho, V, 0_{\mathrm{End}_{\mathbb{C}}(V)})$ is a Weil-Deligne representation.

If we want to construct an example where N is not trivial we will need to use a V with $\dim_{\mathbb{C}}(V) \geq 2$. Let's begin our exploration by constructing an example for $n = 2$.

Let $V = \langle e_1, e_0 \rangle_{\mathbb{C}}$ and define:

$$\rho_2(w) = \begin{pmatrix} |w| & 0 \\ 0 & 1 \end{pmatrix}, \qquad Ne_0 = e_1, \qquad Ne_1 = 0,$$

so that we may write $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is clear that ρ_2 is a representation and we find that:

$$\begin{aligned} \rho_2(w)N &= \begin{pmatrix} 0 & |w| \\ 0 & 0 \end{pmatrix} \\ &= |w| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |w| & 0 \\ 0 & 1 \end{pmatrix} \\ &= |w|N\rho_2(w), \end{aligned}$$

and so (ρ_2, V, N) is a Weil-Deligne representation.

We may in fact repeat this construction for any $n \geq 2$. Let $V = \langle e_{n-1}, \ldots, e_0 \rangle_{\mathbb{C}}$ and define ρ_n and N by:

$$\rho_n(w)e_i = |w|^i e_i, \qquad Ne_i = e_{i+1} \text{ for } 1 \leq i < n-1, \qquad Ne_{n-1} = 0.$$

Then one may check in a similar manner as above that (ρ_n, V, N) is indeed a Weil-Deligne representation, which we shall henceforth denote by $\mathrm{Sp}(n)$.

The following classification is due to Deligne ([8], Proposition 3.1.3).

Lemma 3.9. *Let (ρ, N, V) be a Frobenius semisimple Weil-Deligne representation. Then ρ is indecomposable (cannot be written as a direct sum of two non-zero Weil-Deligne representations) if and only if it is of the form $\rho_0 \otimes Sp(n)$ where ρ_0 is irreducible. Moreover, ρ_0 and $Sp(n)$ are uniquely determined by ρ up to isomorphism.*

Given two Weil-Deligne representations, it is possible to construct a new one by a process analogous to the familiar tensor product construction.

Definition 3.10. Let $r_1 = (\rho_1, V_1, N_1)$ and $r_2 = (\rho_2, V_2, N_2)$ be Weil-Deligne representations. Their *tensor product* $r_1 \otimes r_2 := (\rho, V_1 \otimes V_2, N)$ is the Weil-Deligne representation given by:

$$\rho(w)(v_1 \otimes v_2) = \rho_1(w)v_1 \otimes \rho_2(w)v_2, \qquad N(v_1 \otimes v_2) = N_1 v_1 \otimes v_2 + v_1 \otimes N_2 v_2,$$

for all $w \in W_K, v_1 \in V_1$ and $v_2 \in V_2$.

We also have that $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ arises naturally as the underlying vector space of a Weil-Deligne representation in the following sense.

Definition 3.11. Let $r_1 = (\rho_1, V_1, N_1)$ and $r_2 = (\rho_2, V_2, N_2)$ be Weil-Deligne representations. We define the *Hom-representation of r_1 and r_2* as $\mathbf{Hom}(r_1, r_2) := (\rho, \mathrm{Hom}_{\mathbb{C}}(V_1, V_2), N)$, given by:

$$(\rho(w)f)(v_1) = \rho_2(w)(f(\rho_1(w)^{-1}v_1)), \qquad (Nf)(v_1) = N_2(f(v_1)) - f(N_1(v_1)),$$

for all $w \in W_K, f \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ and $v_1 \in V_1$.

In particular this allows us to define the *contragredient* of a Weil-Deligne representation as the representation $(\rho^\vee, V^\vee, -N^T) := \mathbf{Hom}(\rho, \mathbf{1})$ where $\mathbf{1} : W_K \to K^\times$ denotes the trivial one-dimensional representation.

Now we may define a natural notion of equivalence on Weil-Deligne representations.

Definition 3.12. Let $r_1 = (\rho_1, V_1, N_1)$ and $r_2 = (\rho_2, V_2, N_2)$ be Weil-Deligne representations. We say that r_1 is *equivalent* to r_2 if there exists a linear isomorphism $f : V_1 \to V_2$ such that:

$$f \circ \rho_1 = \rho_2 \circ f \quad \text{and} \quad f \circ N_1 = N_2 \circ f.$$

For the sake of completeness we shall also introduce the notion of the Weil-Deligne group.

Definition 3.13. The *Weil-Deligne group of K* is the group scheme $WD_K = W_K \ltimes \mathbb{G}_a$ over $\mathbb{Q}$, where the action is:

$$wxw^{-1} = |w|x$$

for $w \in W_K, x \in \mathbb{G}_a$. Moreover, the composition is given by:

$$(w_1, x_1) \cdot (w_2, x_2) = (w_1 w_2, |w_2|^{-1} x_1 + x_2).$$

Some references define this side of the bijection based on finite dimensional representations of WD_K. Recall from representation theory that finite dimensional representations of $\mathbb{G}_a$ correspond to nilpotent endomorphisms. It then follows that a Weil-Deligne representation of W_K is the same as a finite dimensional representation of WD_K, and so both of these characterisations are in fact equivalent.

3.3. **Relation to ℓ-adic representations.** With the definition of Weil-Deligne representations and some of their properties in hand, we shall now give some motivation for their introduction in the first place. Most representations of the Weil group W_K which occur in nature are in fact so called 'ℓ-adic representations'. The prototypical examples of these representations are provided by étale cohomology groups, which are the algebraic analogue of cohomology groups in topology. Discussion of étale cohomology is outwith the scope of this essay, but the reader is referred to a text such as [31]. In spite of this we are still able to explore some connections between the two in what follows. This section is significantly lengthier than the others in order to allow ample room for thorough explanation of the results encountered and to emphasise the importance of understanding this side of the bijection in this way.

We begin by recalling the definition of an ℓ-adic representation.

Definition 3.14. (ℓ-adic representation of W_K) Let $\ell \neq p$ be a prime number. An *ℓ-adic representation of W_K* is a continuous (relative to the ℓ-adic topology on $GL_n(\overline{\mathbb{Q}_\ell})$) homomorphism $\rho : W_K \to GL_n(\overline{\mathbb{Q}_\ell})$.

Recall that the *tame inertia* I_K/P_K is isomorphic to $\prod_{\ell \neq p} \mathbb{Z}_\ell$, where the isomorphism is uniquely determined up to multiplication by an element of $\prod_{\ell \neq p} \mathbb{Z}_\ell^\times$. In particular, for each $\ell \neq p$ there is a continuous surjection $t_\ell : I_K \twoheadrightarrow \mathbb{Z}_\ell$ given by:

$$t_\ell : I_K \twoheadrightarrow I_K/P_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \twoheadrightarrow \mathbb{Z}_\ell.$$

which realises $\mathbb{Z}_\ell$ as the maximal pro-ℓ quotient of I_K. It is also the case that the kernel of t_ℓ fits into an exact sequence:

$$0 \to P_K \to \ker(t_\ell) \to \prod_{q \neq \ell, p} \mathbb{Z}_q \to 0,$$

and it therefore follows that the pro-order of $\ker(t_\ell)$ is coprime to ℓ.

By tracking through the group action through the aforementioned isomorphism we arrive at the following result.

Lemma 3.15. *For all $w \in W_K, \sigma \in I_K$ we have:*

$$t_\ell(w\sigma w^{-1}) = |w| t_\ell(\sigma).$$

We may now introduce the following classical result of Grothendieck. Our proof shall follow the one given in ([5], Theorem 32.6). An alternative, more classical proof may be found in the appendix of [39].

Theorem 3.16. *(Grothendieck's ℓ-adic Monodromy Theorem) Let (ρ, V) be an ℓ-adic representation of W_K. Then there exists a unique nilpotent endomorphism $N \in End_{\overline{\mathbb{Q}_\ell}}(V)$ such that for all h in a sufficiently small open subgroup H of I_K, we have:*

$$\rho(h) = \exp(t_\ell(h)N).$$

Proof. To begin with we shall show uniqueness. Suppose that there exists an $N \in \mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ and an open subgroup H of I_K such that the statement holds, and fix some $h \in H$ such that $t_\ell(h) \neq 0$. In that case we have:

$$\begin{aligned} & \exp(t_\ell(h)N) = \rho(h) \\ \Longrightarrow \quad & t_\ell(h)N = \log(\rho(h)) \\ \Longrightarrow \quad & N = \frac{1}{t_\ell(h)} \log(\rho(h)), \end{aligned}$$

and so uniqueness is immediate.

Let $\mathcal{O}_\ell$ denote the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}_\ell}$. We define a filtration on $GL_n(\mathcal{O}_\ell)$ by:

$$G_k := \{g \in GL_n(\mathcal{O}_\ell) \mid g \equiv I_n \bmod \ell^k\}.$$

Each of these are an open subgroup of $GL_n(\mathcal{O}_\ell)$, and moreover for all $k \in \mathbb{N}$ we have an isomorphism:

$$G_k/G_{k+1} \xrightarrow{\sim} M_n(\mathcal{O}_\ell/\ell\mathcal{O}_\ell),$$

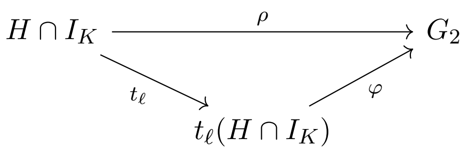
which is defined by sending $(I_n + \ell^k g)G_{k+1} \mapsto g$. This is then a finite group of exponent ℓ. Now we define an open subgroup of $\ker(t_\ell)$ by:

$$J := \{g \in \ker(t_\ell) \mid \rho(g) \in G_2\}.$$

Consider the image of $\rho(J)$ in G_2/G_3 after applying the quotient map. Since $\ker(t_\ell)$ has pro-order coprime to ℓ and G_2/G_3 is a group of exponent ℓ, we must have that this is trivial, and so $\rho(J) \subseteq G_3$. We may then repeat this argument inductively to see that $\rho(J) \subseteq G_k$ for all $k \geq 2$, and so we deduce that $\rho(J)$ is trivial.

Since J is open in $\ker(t_\ell)$ we may find an open subgroup H' of I_K with $H' \cap \ker(t_\ell) \subseteq J$. We choose to shrink H' if necessary so that $\rho(H') \subseteq G_2$. Similarly we may also find an open normal subgroup H of W_K such that $H \cap I_K \subseteq H'$.

Consider the restriction of ρ to $H \cap I_K$. It follows that there is a continuous homomorphism $\varphi : t_\ell(H \cap I_K) \to G_2$ which this restriction factors through, so that the following diagram commutes:



By Lemma 3.15 we know that:

$$t_\ell(\Phi\sigma\Phi^{-1}) = |\Phi| t_\ell(\Phi).$$

Applying φ to both sides and using that $|\Phi| = q^{-1}$, we have:

$$\rho(\Phi h \Phi^{-1})^q = \rho(h).$$

Thus $\rho(h)^q$ and $\rho(h)$ are conjugate, and they must therefore have the same set of eigenvalues.

In particular, it follows that if λ is an eigenvalue of $\rho(h)$, then λ^q is an eigenvalue of $\rho(h)^q$. Since $\rho(h)$ is invertible, we must then have that $\lambda \neq 0$ and so the eigenvalues must consist solely of q^{th} roots of unity.

Since $\rho(H) \subseteq G_2$ we must have that $\lambda \in 1 + \ell^2\mathcal{O}_\ell$. This is in fact torsion free - to see this we note that $1 + \ell^2\mathcal{O}_\ell$ isomorphic to $\ell^2\mathcal{O}_\ell$ via the logarithm map, which is certainly torsion free. It follows that all of the eigenvalues are equal to 1 and so $\rho(h)$ is unipotent for every $h \in H$.

Now we may argue existence. Choose some $h_0 \in H \cap I_K$ with $t_\ell(h_0) \neq 0$ and as before set:

$$N := \frac{1}{t_\ell(h_0)} \log(\rho(h_0)).$$

We claim that this choice of N works. Since $\rho(h_0)$ is unipotent, $\log(\rho(h_0))$ is therefore well-defined and nilpotent, so N is nilpotent. In that case:

$$\rho(h_0) = \exp(t_\ell(h_0)N).$$

We wish to now extend this to an open subgroup of I_K. To do so, we notice that the two group homomorphisms $t_\ell(H \cap I_K) \to GL_n(\overline{\mathbb{Q}_\ell})$ given by $x \mapsto \varphi(x)$ and $x \mapsto \exp(xN)$ agree on $t_\ell(h_0)$:

$$\begin{aligned}\varphi(t_\ell(h_0)) &= \rho(h_0)\\ &= \exp(\log(\rho(h_0)))\\ &= \exp\left(t_\ell(h_0)\cdot\frac{1}{t_\ell(h_0)}\log(\rho(h_0))\right)\\ &= \exp(t_\ell(h_0)N).\end{aligned}$$

By continuity they must also agree on the closure $\mathbb{Z}_\ell t_\ell(h_0)$ of $\mathbb{Z}_\ell$. Define $H := t_\ell^{-1}(\mathbb{Z}_\ell t_\ell(h_0))$. Observe that $\mathbb{Z}_\ell t_\ell(h_0)$ is open in $\mathbb{Z}_\ell$ since $t_\ell(h_0) \neq 0$. It follows that H is an open subgroup of I_K since t_ℓ is continuous, and so $\rho(h) = \varphi(t_\ell(h)) = \exp(t_\ell(h)N)$ for all $h \in H$ as required. □

Theorem 3.16 is incredibly useful as it now allows us to relate ℓ-adic representations with Weil-Deligne representations. To formalise this, we shall prove the following theorem following [9].

Before we do so, we need a small result which will help us.

Lemma 3.17. *Let ρ and N be as in Theorem 3.16. Then for all $w \in W_K$ we have:*

$$\rho(w)N\rho(w)^{-1} = |w|N.$$

Proof. Let H be chosen in such a way that we may apply Theorem 3.16, and such that it is normal in W_K. Then for $h \in H$ and for all $w \in W_K$ we can use Lemma 3.15 to see that:

$$\begin{aligned}\rho(whw^{-1}) &= \exp(t_\ell(whw^{-1})N)\\ &= \exp(t_\ell(h)|w|N),\end{aligned}$$

On the other hand, using the fact that ρ is a homomorphism:

$$\begin{aligned}\rho(whw^{-1}) &= \rho(w)\rho(h)\rho(w)^{-1}\\ &= \rho(w)\exp(t_\ell(h)N))\rho(w)^{-1}\\ &= \exp(t_\ell(h)\rho(w)N\rho(w)^{-1}).\end{aligned}$$

The claim then follows from the injectivity of exp. □

Once again, our proof of the following result follows the one presented in [5].

Theorem 3.18. *Fix a lift $\Phi \in W_K$ of the Frobenius map and a continuous surjective map $t_\ell : I_K \twoheadrightarrow \mathbb{Z}_\ell$. Then there is an equivalence of categories:*

$$\left\{\begin{array}{c}\ell\text{-adic representations}\\ \text{of } W_K\end{array}\right\} \xrightarrow{WD_{\Phi,t_\ell}} \left\{\begin{array}{c}\text{Weil-Deligne representations}\\ \text{of } W_K \text{ over } \overline{\mathbb{Q}_\ell}\end{array}\right\}.$$

Moreover, up to a natural isomorphism the functor WD_{Φ,t_ℓ} is independent of the choices of Φ and t_ℓ.

Proof. Let $\rho : W_K \to GL_n(V)$ be an ℓ-adic representation of W_K. By Theorem 3.16, there exists a nilpotent $N \in \mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V)$ such that:

$$\rho(h) = \exp(t_\ell(h)N)$$

for all h in an open subgroup H of I_K. We now define a map $\rho_\Phi : W_K \to GL_n(V)$ by:

$$\rho_\Phi(\Phi^n\sigma) := \rho(\Phi^n\sigma)\exp(-t_\ell(\sigma)N),$$

for all $n \in \mathbb{Z}$ and all $\sigma \in I_K$. We claim that $\mathrm{WD}_{\Phi,t_\ell}(\rho, V) := (\rho_\Phi, V, N)$ is a Weil-Deligne representation. Firstly, by Lemma 3.17 it follows that for all $w \in W_K$:

$$\begin{aligned}\rho_\Phi(w)N\rho_\Phi(w)^{-1} &= \rho(w)\exp(-t_\ell(w)N)N\exp(t_\ell(w)N)\rho(w)^{-1}\\ &= \rho(w)N\rho(w)^{-1}\\ &= |w|N,\end{aligned}$$

where we use that the inner exponentials commute with N as all of their terms are multiples of N. We now claim that ρ_Φ is a group homomorphism. Indeed, using Lemma 3.17 we may verify that for all $\sigma_1, \sigma_2 \in I_K$ and $n_1, n_2 \in \mathbb{Z}$, we have:

$$\begin{aligned}
\rho_\Phi(\Phi^{n_1}\sigma_1\Phi^{n_2}\sigma_2) &= \rho_\Phi(\Phi^{n_1+n_2}(\Phi^{-n_2}\sigma_1\Phi^{n_2}\sigma_2)) \\
&= \rho(\Phi^{n_1}\sigma_1\Phi^{n_2}\sigma_2)\exp(-t_\ell(\Phi^{-n_2}\sigma_1\Phi^{n_2}\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1\Phi^{n_2}\sigma_2)\exp(-(t_\ell(\Phi^{-n_2}\sigma_1\Phi^{n_2}) + t_\ell(\sigma_2))N) \\
&= \rho(\Phi^{n_1}\sigma_1)\rho(\Phi^{n_2}\sigma_2)\exp(-t_\ell(\Phi^{-n_2}\sigma_1\Phi^{n_2})N)\exp(-t_\ell(\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1)\rho(\Phi^{n_2}\sigma_2)\exp(-|\Phi^{-n_2}|t_\ell(\sigma_1)N)\exp(-t_\ell(\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1)\rho(\Phi^{n_2}\sigma_2)\exp(-|\Phi^{-n_2}|t_\ell(\sigma_1)N)\rho(-\Phi^{n_2}\sigma_2)\rho(\Phi^{n_2}\sigma_2)\exp(-t_\ell(\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1)\exp(-|\Phi^{-n_2}|t_\ell(\sigma_1)\rho(\Phi^{n_2}\sigma_2)N\rho(-\Phi^{n_2}\sigma_2))\rho(\Phi^{n_2}\sigma_2)\exp(-t_\ell(\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1)\exp(-|\Phi^{-n_2}|t_\ell(\sigma_1)|\Phi^{n_2}\sigma_2|N)\rho(\Phi^{n_2}\sigma_2)\exp(-t_\ell(\sigma_2)N) \\
&= \rho(\Phi^{n_1}\sigma_1)\exp(-t_\ell(\sigma_1)N)\rho(\Phi^{n_2}\sigma_2)\exp(-t_\ell(\sigma_2)N) \\
&= \rho_\Phi(\Phi^{n_1}\sigma_1)\rho_\Phi(\Phi^{n_2}\sigma_2),
\end{aligned}$$

as required, where we used that $|\sigma_2| = 1$ since $\sigma_2 \in I_K$. Moreover, by definition of N we find that ρ_Φ is trivial on H, so ρ_Φ has open kernel and is therefore continuous. It is therefore a valid Weil-Deligne representation.

Now let $\varphi : (\rho_1, V_1) \to (\rho_2, V_2)$ be a map of representations, i.e a map $\varphi : V_1 \to V_2$ such that:

$$\varphi \circ \rho_1 = \rho_2 \circ \varphi.$$

We claim that φ then gives rise to a morphism of Weil-Deligne representations $(\rho^1_\Phi, V_1, N_1) \to (\rho^2_\Phi, V_2, N_2)$, where $\rho^1_\Phi(\Phi^n\sigma) = \rho_1(\Phi^n\sigma)\exp(-t_\ell(\sigma)N)$ and similarly for ρ^2_Φ.

Firstly we must show that $\varphi \circ N_1 = N_2 \circ \varphi$. This follows immediately since we know that if $h_0 \in H$ is some element with $t_\ell(h_0) \neq 0$ then:

$$N_i = \frac{1}{t_\ell(h_0)}\log(\rho_i(h_0)), \quad i \in \{1, 2\},$$

and so the result follows. Now observe that for all $w \in W_K$ we have:

$$\begin{aligned}
\varphi \circ \rho^1_\Phi(w) &= \varphi\rho_1(w)\exp(-t_\ell(w)N_1) \\
&= \rho_2(w)\varphi\exp(-t_\ell(w)N_1) \\
&= \rho_2(w)\exp(-t_\ell(w)N_2)\varphi \\
&= \rho^2_\Phi(w) \circ \varphi.
\end{aligned}$$

Now if (ρ_Φ, V, N) is a Weil-Deligne representation then it is straightforward to verify that for $n \in \mathbb{Z}, \sigma \in I_K$, the map:

$$\rho(\Phi^n\sigma) := \rho_\Phi(\Phi^n\sigma)\exp(t_\ell(\sigma)N),$$

gives rise to an ℓ-adic representation $\rho : W_K \to GL(V)$ which then yields:

$$WD_{\Phi,t_\ell}((\rho, V)) = (\rho_\Phi, V, N).$$

Therefore we have an equivalence of categories.

To finish the proof we must demonstrate that if (Φ', t'_ℓ) was chosen instead of (Φ, t_ℓ) then the two functors $\mathrm{WD}_{\Phi,t_\ell}$ and $\mathrm{WD}_{\Phi',t'_\ell}$ which we constructed are naturally isomorphic. We will break this step down into proving that $WD_{\Phi,t_\ell} \cong WD_{\Phi',t_\ell}$ and that $WD_{\Phi,t_\ell} \cong WD_{\Phi,t'_\ell}$.

We fix $\alpha \in I_K$ such that $\Phi = \Phi'\alpha$. Let (ρ, V) be an ℓ-adic representation and define $(r_1, V, N) := WD_{\Phi,t_\ell}((\rho, V))$ and $(r_2, V, N) := WD_{\Phi',t_\ell}((\rho, V))$. We must find a linear isomorphism $f : V \to V$ such that $f \circ N = N \circ f$ to prove that these representations are equivalent.

To do so, we try choosing f in the form $f = \exp(\lambda N)$ where λ is a scalar. Now for $w \in W_K$, we may write it in the form $w = \Phi x$ for $x \in I_K$. Then:

$$f \circ r_1(w) = r_2(w) \circ f \iff f \circ r_1(\Phi) = r_2(\Phi) \circ f \text{ and } f \circ r_1(x) = r_2(x) \circ f$$

By definition we know that $r_1(x) = r_2(x) = \rho(x)\exp(-t_\ell(x)N)$ for $x \in I_K$ and that $r_1(\Phi) = \rho(\Phi)$, whereas on the other hand:

$$r_2(\Phi) = r_2(\Phi'\alpha) = \rho(\Phi)\exp(-t_\ell(\alpha)N).$$

From Lemma 3.17 we have that if $x \in I_K$ then:

$$\rho(x)N = N\rho(x).$$

Since $\rho(x)$ commutes with N we therefore have that:

$$\begin{aligned} f \circ r_1(x) &= \exp(\lambda N)\rho(x)\exp(-t_\ell(x)N) \\ &= \rho(x)\exp(\lambda N)\exp(-t_\ell(x)N) \\ &= \rho(x)\exp(-t_\ell(x)N)\exp(\lambda N) \\ &= r_2(x) \circ f. \end{aligned}$$

It will now suffice to find a λ which satisfies the other condition, i.e such that:

$$\exp(\lambda N)\rho(\Phi) = \rho(\Phi)\exp(-t_\ell(x)N)\exp(\lambda N).$$

We can use Lemma 3.17 to reduce this as follows:

$$\begin{aligned} & \exp(\lambda N)\rho(\Phi) = \rho(\Phi)\exp(-t_\ell(x)N)\exp(\lambda N) \\ \implies \quad & \rho(\Phi)^{-1}\exp(\lambda N)\rho(\Phi) = \exp((\lambda - t_\ell(\alpha))N) \\ \implies \quad & \rho(\Phi^{-1})\exp(\lambda N)\rho(\Phi^{-1})^{-1} = \exp((\lambda - t_\ell(\alpha))N) \\ \implies \quad & \exp(|\Phi|^{-1}\lambda N) = \exp((\lambda - t_\ell(\alpha))N). \end{aligned}$$

Recalling that $|\Phi| = q^{-1}$, it follows from the injectivity of exp that we must then have:

$$\lambda = \frac{t_\ell(\alpha)}{1-q}.$$

This works and so we have that $\mathrm{WD}_{\Phi,t_\ell} \cong \mathrm{WD}_{\Phi',t_\ell}$.

Now we let t'_ℓ be another choice of t_ℓ, so that $t'_\ell = yt_\ell$ for some $y \in \mathbb{Z}_\ell^\times$. Then we have that:

$$\rho(h) = \exp(t_\ell(h)N) = \exp(t'_\ell(h)y^{-1}N).$$

This says that if $\mathrm{WD}_{\Phi,t_\ell}((\rho,V)) = (r,V,N)$ then $\mathrm{WD}_{\Phi,t'_\ell}((\rho,V)) = (r,V,y^{-1}N)$. It will therefore suffice to find a linear isomorphism $f : V \to V$ such that f commutes with $r(W_K)$, and also $yf \circ N = N \circ f$. Fix a $m \in \mathbb{N}$ such that $r(\Phi^m)$ is central in $r(W_K)$.

We may then decompose V as a direct sum of generalized eigenspaces of $r(\Phi^m)$:

$$V = \bigoplus_\lambda \ker(r(\Phi^m) - \lambda \mathrm{id}_V)^{\dim V} := \bigoplus_\lambda V_\lambda.$$

On each V_λ we define f to be multiplication by a scalar $a_\lambda \in K^\times$. By our choice of m, each space V_λ is stable under $r(W_K)$ and so we have that f commutes with $r(W_K)$.

Let $v \in E_\lambda$. Using Lemma 3.17 we have that $r(\Phi^m)N = |\Phi^m|Nr(\Phi^m)$ and so:

$$\begin{aligned} r(\Phi^m)Nv &= |\Phi^m|Nr(\Phi^m)v \\ &= |\Phi^m|N\lambda v \\ &= |\Phi^m|\lambda Nv, \end{aligned}$$

so that $NE_\lambda \subseteq E_{|\Phi^m|\lambda}$.

Since the subscripts satisfy a linear relation, we may choose the $a_\lambda \in K^\times$ such that they satisfy $a_\lambda = y a_{|\Phi^m|\lambda}$. In that case, f yields an isomorphism $\mathrm{WD}_{\Phi,t_\ell}((\rho, V)) \cong \mathrm{WD}_{\Phi,t'_\ell}((\rho, V))$. We may define such an f for any (ρ, V) and so $\mathrm{WD}_{\Phi,t_\ell} \cong \mathrm{WD}_{\Phi,t'_\ell}$.

Putting these results together we have that:

$$\mathrm{WD}_{\Phi,t_\ell} \cong \mathrm{WD}_{\Phi',t_\ell} \cong \mathrm{WD}_{\Phi',t'_\ell},$$

as required. □

Remark 3.19. Theorem 3.18 gives us a canonical equivalence between the aforementioned categories. Moreover if we choose an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ we induce a further equivalence with the category of complex Weil-Deligne representations of W_K.

As a consequence of this theorem we may give the following preliminary definition which we soon refine, following [5], which explains where the term 'Frobenius' in 'Frobenius semisimple' arises from.

Definition 3.20. Let (ρ, V) be an ℓ-adic representation of W_K. We say that (ρ, V) is Φ-*semisimple* if the associated Weil-Deligne representation $\mathrm{WD}_{\Phi,t_\ell}(\rho, V)$ is semisimple.

Lemma 3.21. *Let (ρ, V) be an continuous finite-dimensional complex representation of W_K and $\Phi \in W_K$ any lift of the geometric Frobenius. Then the representation ρ is semisimple if and only if the automorphism $\rho(\Phi)$ is semisimple.*

Proof. See ([5], Proposition 28.7). □

Proposition 3.22. *Let (ρ, V) be an ℓ-adic representation of W_K. The following conditions are equivalent:*

(a) (ρ, V) is Φ-semisimple.
(b) There is a lift of the geometric Frobenius $\Phi \in W_K$ such that $\rho(\Phi)$ is semisimple.
(c) For all $w \in W_K \backslash I_K$, $\rho(w)$ is semisimple.

Proof. Since $\rho_\Phi(\Phi) = \rho(\Phi)$, it follows from Lemma 3.21 that $(a) \Leftrightarrow (b)$.

It is clear that $(c) \implies (b)$. Conversely, let $w \in W_K \backslash I_K$. If w can be written in the form $w = \Phi\alpha$ for $\alpha \in I_K$, then similar to the proof of Theorem 3.18 we have that $\rho(\Phi)$ is conjugate to $\rho(w)$, and so $\rho(w)$ is also semisimple. In the general case where $w = \Phi^n\alpha$ for some $\alpha \in I_K$ we may restrict to W_F for F/K the unramified extension of degree $|n|$, in which case the same argument will hold. □

In light of Proposition 3.22 and the result that $\mathrm{WD}_{\Phi,t_\ell} \cong \mathrm{WD}_{\Phi',t_\ell}$ we may refine our definition as follows.

Definition 3.23. Let (ρ, V) be an ℓ-adic representation of W_K. We say that (ρ, V) is *Frobenius semisimple* if (ρ, V) is Φ-semisimple for some lift of the geometric Frobenius $\Phi \in W_K$.

The resulting consequence of this result is the following classification.

Theorem 3.24. *There is a canonical equivalence of categories:*

$$\left\{ \begin{array}{c} n\text{-dimensional Frobenius} \\ \text{semisimple } \ell\text{-adic} \\ \text{representations of } W_K \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} n\text{-dimensional Frobenius} \\ \text{semisimple Weil-Deligne} \\ \text{representations of } W_K \text{ over } \overline{\mathbb{Q}_\ell} \end{array} \right\}.$$

Furthermore, the choice of an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$ induces a further equivalence with:

$$\left\{ \begin{array}{c} n\text{-dimensional Frobenius} \\ \text{semisimple Weil-Deligne} \\ \text{representations of } W_K \text{ over } \mathbb{C} \end{array} \right\},$$

which is the term appearing in Theorem 1.3.

3.4. $L+\epsilon$ **factors.** We shall now introduce the L and ϵ factors on this side of the bijection, the construction of which differ to the factors we defined in section 2.4. Much of the theory presented here is summarised in [25] and [42], and we shall closely follow the order of exposition given in [7] and [45]. Let (ρ, V, N) be a Frobenius semisimple n-dimensional complex representation of W_K and fix a lift of the geometric Frobenius $\Phi \in W_K$. Write $V_N := \ker(N)$ and let:

$$V_N^{I_K} = \{v \in V_N : \rho(x)v = v \text{ for all } x \in I_K\}$$

denote the I_K-invariants in V_N. Note that $\rho(\Phi)$ is well defined as a linear map on $V_N^{I_K}$ and so we may make the following definition.

Definition 3.25. The *L-factor* of (ρ, V, N) is given by:

$$L(\rho, s) := \det\left(1 - q^{-s}\rho(\Phi)\mid_{V_N^{I_K}}\right)^{-1}.$$

L-factors have the property of being multiplicative. That is to say, if $0 \to \rho' \to \rho \to \rho'' \to 0$ is an exact sequence of such representations, then:

$$L(\rho, s) = L(\rho', s)L(\rho'', s).$$

Equivalently we also have that:

$$L(\rho' \otimes \rho'', s) = L(\rho', s)L(\rho'', s).$$

They are also inductive, in the sense described in ([42], Definition 2.3.2).

If ρ is one-dimensional we may use the Artin map to identify it with a character of W_K^{ab}, which we will write this as $\chi := \text{Art}_K^{-1}(\rho)$. We now introduce the two cases which concern us.

Definition 3.26. We say that a character $\chi : W_K^{\text{ab}} \to \mathbb{C}^\times$ is *unramified* if $\chi(I_K) = \{1\}$. Otherwise we say that χ is *ramified*.

On this side of the bijection the ϵ-factors are considerably more difficult to define. We shall begin by defining them for one-dimensional representations and then use results of Deligne from [9].

Definition 3.27. The *exponent* of a character ψ is:

$$n(\psi) := \max\{n \in \mathbb{Z} : \psi(\pi_K^{-n}\mathcal{O}_K) = \{1\}\}.$$

We have a similar notion of the conductor (Definition 2.32) on this side of the bijection.

Definition 3.28. The *conductor* $f(\chi)$ of χ is defined to be 0 if χ is unramified, and:

$$f(\chi) := \min\{n \in \mathbb{N}_0 : \chi(\text{Art}_K(1 + \pi_K^n\mathcal{O}_K)) = \{1\}\},$$

if χ is ramified.

This then allows us to introduce the following ϵ-functions in both cases.

Definition 3.29. Let dx be an additive Haar measure of K and ψ a non-trivial additive character. Suppose that χ is unramified and let $x \in W_K$ be such that $|x| = q^{-n(\psi)}$. Define:

$$\epsilon(\chi, \psi, dx) := \frac{\chi(x)}{|x|}\int_{\mathcal{O}_K} dx.$$

If χ is ramified we can make a similar definition.

Definition 3.30. Let dx be an additive Haar measure of K and ψ a non-trivial additive character. Suppose that χ is ramified and let $x \in W_K$ be such that $|x| = q^{-(n(\psi)+f(\chi))}$. Define:

$$\epsilon(\chi, \psi, dx) = \int_{x^{-1}\mathcal{O}_K^\times} \chi(\text{Art}_K(x))\psi(x)\,dx.$$

To generalise these to higher dimensions we shall use an existence theorem given by Deligne in [9]. Currently there is no known explicit formula for the ϵ-functions, which isn't surprising given that everything hinges on the isomorphism of Theorem 1.1. To understand this result we must first introduce the Grothendieck group.

Definition 3.31. The *Grothendieck group* $R(W_K)$ of W_K is the free abelian group on the isomorphism classes of continuous finite dimensional irreducible representations of W_K. If ρ is a representation of W_K we denote by $[\rho]$ the image of ρ in $R(W_K)$. Furthermore, we define the *degree* of an element $n_1[\rho_1] + \cdots + n_r[\rho_r] \in R(W_K)$ to be $n_1 + \cdots + n_r$.

Theorem 3.32. *(Deligne) There is a unique ϵ-function which associates with each choice of a non-trivial additive character ψ of K, an additive Haar measure dx on K and a representation ρ of W_K a complex number $\epsilon(\rho, \psi, dx) \in \mathbb{C}^\times$ such that:*

- *(a) $\epsilon(\rho, \psi, dx) = \epsilon(\chi, \psi, dx)$ as defined above if ρ is a representation of degree 1 corresponding to a quasi-character χ,*
- *(b) $\epsilon(\cdot, \psi, dx)$ is multiplicative in exact sequences of representations of W_K.*
- *(c) For every tower of finite extensions $L'/L/K$ and for every choice of additive Haar measures μ_L on L and $\mu_{L'}$ on L' we have:*

$$\epsilon(Ind_{L'/L}[\rho'], \psi \circ Tr_{L/K}, \mu_L) = \epsilon([\rho'], \psi \circ Tr_{L'/K}, \mu_{L'}),$$

for $[r'] \in R(W_K)$ with degree 0.

With this result in hand we may define the ϵ-factor of the Weil Deligne representation (ρ, V, N) as follows.

Definition 3.33. Let dx be the Haar measure on K which is self-dual with respect to the Fourier transform $f \mapsto \hat{f}$ defined by ψ:

$$\hat{f}(y) = \int f(x)\psi(xy)\, dx.$$

Define the *ϵ-factor associated to the Weil-Deligne representation* (ρ, V, N) by:

$$\epsilon(\rho, \psi, s) := \epsilon(|\cdot|^s \rho, \psi, dx) \cdot \det\left(-\rho(\Phi)_{V^{I_K}/V_N^{I_K}}\right).$$

Remark 3.34. It is true ([42], Section 3.4) that for the above choice of dx we have that $\mathrm{vol}_{\mathrm{dx}}(\mathcal{O}_K) = q^{-d/2}$, where d denotes the valuation of the absolute different of K.

We then have a similar result to Theorem 2.33:

Proposition 3.35. *Let ρ be an irreducible Weil-Deligne representation of dimension n. Then:*

$$\epsilon(\rho, \psi, s) = \epsilon(\rho, \psi, 0)q^{-s(f(\rho)+n\cdot n(\psi))}.$$

Example 3.36. We now calculate L and ϵ-factors for the representation $\mathrm{Sp}(n)$ from Example 3.8. In that case, by definition we have that $V_N = \mathbb{C}e_{n-1}$ and clearly $V_N^{I_K} = \mathbb{C}e_{n-1}$ as well. Furthermore, for a fixed lift of the geometric Frobenius $\Phi \in W_K$ we have that $\Phi e_i = q^{-i}e_i$. It follows that the L-factor is given by:

$$L(\rho, s) = \frac{1}{1 - q^{1-s-n}}.$$

Now choose an additive character ψ with exponent 0 and let dx be the Haar measure defined in Definition 3.33. Then we have that $\epsilon(r, \psi, dx) = q^{-nd/2}$ where d is again the valuation of the absolute different of K. It follows that the ϵ-factor is given by:

$$\epsilon(\rho, \psi, s) = (-1)^{n-1}q^{\frac{-nd-(n-1)(n-2)}{2}}.$$

4. Conclusion

We now have all the tools to hand to understand the statement of Theorem 1.3. The statement is remarkably powerful, giving rise to local class field theory in the case where $n = 1$ and a plethora of other results which are discussed in [16]. We now briefly mention another consequence of this result which is treated formally in [15].

Similar to Definition 3.26, we say that a Weil-Deligne representation is *unramified* if it is trivial on the inertia subgroup I_K. Upon restricting the bijection in Theorem 1.3 to the unramified Weil-Deligne representations on one side, we find that the corresponding representations on the GL_n side are the so-called *spherical representations of* $GL_n(K)$, which are representations (ρ, V) of $GL_n(K)$ for which $V^{GL_n(\mathcal{O}_K)} \neq 0$.

$$\left\{ \begin{array}{c} \text{Representations } (\rho, V) \\ \text{of } GL_n(K) \text{ for} \\ \text{which } V^{GL_n(\mathcal{O}_K)} \neq \{0\} \end{array} \right\}_{/\cong} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Weil-Deligne representations} \\ \text{such that } N = 0 \text{ and} \\ \rho : W_K \twoheadrightarrow \mathbb{Z} \hookrightarrow GL_n(\mathbb{C}) \end{array} \right\}_{/\cong}$$

Furthermore, it is a fact that the semisimple representations of W_K which factor through $\mathbb{Z}$ correspond to semisimple elements of $GL_n(\mathbb{C})$ (i.e those which are diagonalisable). Similarly, irreducible representations (ρ, V) of $GL_n(K)$ for which $V^{GL_n(\mathcal{O}_K)} \neq \{0\}$ correspond to irreducible representations of the so called "spherical Hecke algebra", $\mathcal{H}_{\text{sph}}(K)$. Restricting to these cases we therefore obtain a bijection:

$$\left\{ \begin{array}{c} \text{Irreducible representations} \\ \text{of } \mathcal{H}_{\text{sph}}(K) \end{array} \right\}_{/\cong} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Semisimple elements} \\ \text{in } GL_n(\mathbb{C}) \end{array} \right\}_{/\cong}$$

This is known as the *Satake isomorphism*, which was introduced by Satake in [36]. It therefore happens that, surprisingly, the left-hand side of this bijection is completely independent of K.

One of the most prominent open problems in the area related to the local Langlands correspondence for GL_n is the problem of finding a purely local proof. All of the known proofs today rely on an understanding of the ℓ-adic cohomology of a certain family of Shimura varieties, which are global in nature. A few entries in the dictionary and some problems are described by Harris in [16], in which he also details the open problems concerning cohomological realizations of the local correspondence and finding an explicit parametrization of supercuspidal representations.

A similar version of the local Langlands correspondence also holds for arbitrary reductive groups over non-archimedean local fields. In general it is not known how best to state the conjectures in this more general case. Some results are known for classical groups - for example, a version of the correspondence was proven by Gan and Takeda for the symplectic similitude group $GSp(4)$ in [14] and also for the symplectic group $Sp(4)$ in [13].

Finally, a geometric reformulation of the local Langlands correspondence has been developed by Fargues and Scholze in [11]. They claim that it may be recast as an equivalence of categories between the derived category of certain coherent sheaves on the moduli stack of L-parameters and the derived category of certain ℓ-adic sheaves on the moduli stack of G-torsors on the Fargues-Fontaine curve. More recently, further work has been completed in this area by Anschütz and Bras in [1], in which they prove various results related to Fargues-Scholze parameters and verify the existence of the Hecke eigensheaf.

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