Classifying toposes

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Introduction

Algebraic theories can be interpreted 'over' a topological space. For example, an interpretation of the theory of groups over a topological space X assigns a group $G(U)$ to each open set U of X, in a suitably compatible way: for instance, if $V \subseteq U$, then there is a canonical homomorphism $G(U) \to G(V)$. Such a collection of groups, called a *sheaf*, can be thought of as encapsulating different states of knowledge about a single object G, which we can analyse in terms of the sentences that it satisfies. Any given sentence in the language of groups is classically either true or false for each $G(U)$. Analogously to Heyting semantics, the truth value of a sentence φ interpreted in G will be an open set of X, namely, the union of the open sets U for which φ holds in $G(U)$. For example, the group axioms will take the maximal truth value X, so in this sense G 'is' a group. We will show that this method of reasoning is logically sound, provided that we only use a constructive form of logic, and that the sentences in question are suitably *geometric*. In this essay, we will study the idea of interpreting theories over spaces in detail.

In § 1, we will introduce *Grothendieck toposes*, which are categories of sheaves over some topological space, or possibly a generalised space called a *site*. They have many of the familiar properties from the usual category of sets, such as small limits and colimits, exponentials, power objects, and image factorisations. These constructions will allow us in § 2 to interpret arbitrary *geometric theories* in Grothendieck toposes, or equivalently, 'over' the underlying site.

In § 3, we will study *geometric logic*. This is a form of many-sorted first-order logic that has nice properties when used with Grothendieck toposes. It is sound and complete, in the sense that the provable consequences of a geometric theory $\mathbb{T}$ are precisely those that hold in all of its models in Grothendieck toposes. We prove the completeness theorem for geometric logic using a similar method to Gödel's completeness theorem for classical first-order logic: by building a model out of syntax. The usual proof of the latter involves repeatedly extending a theory to a maximal consistent set of sentences; this is required because sentences in such a model must either be true or false. We will not need to do this for geometric logic, since we can build a Grothendieck topos with almost arbitrary truth values. In particular, in § 4 we will build a *universal model* of $\mathbb{T}$ in a particular Grothendieck topos $\mathbf{Set}[\mathbb{T}]$, and show that it satisfies precisely the provable consequences of $\mathbb{T}$. This will not require any use of the axiom of choice.

General principles of category theory suggest that we should also study morphisms between Grothendieck toposes. The relevant notion of morphism for this essay is the *geometric morphism*. If $\mathcal{E}, \mathcal{F}$ are the categories of sheaves over topological spaces X, Y, then, under mild conditions, the geometric morphisms $\mathcal{E} \to \mathcal{F}$ correspond precisely to the continuous functions $X \to Y$. A geometric morphism $\mathcal{E} \to \mathcal{F}$ gives rise to an *inverse image* map that sends models of $\mathbb{T}$ in $\mathcal{F}$ to models of $\mathbb{T}$ in $\mathcal{E}$. The main theorem of this essay, which we will establish in § 4.3, states that every model of $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ arises as a result of applying an inverse image map to the universal model of $\mathbb{T}$. In other words, the geometric morphisms $\mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ correspond to $\mathbb{T}$-models in $\mathcal{E}$. We say that $\mathbf{Set}[\mathbb{T}]$ is the *classifying topos* of the theory $\mathbb{T}$.

A standard reference for introductory topos theory, used extensively in this essay, is Mac Lane and Moerdijk's *Sheaves in Geometry and Logic* [4]. A more comprehensive text is Johnstone's *Sketches of an Elephant* [2], which we reference heavily in §§ 3 and 4. Elementary category theory will be assumed, but some definitions and results will be explicitly referenced from Mac Lane's *Categories for the Working Mathematician* [3], which remains a useful introduction to the subject.

A table of notation used is presented at the end of the essay. We will purposefully work with ambiguous set-theoretic foundations, and will frequently ignore the question of whether classes are proper. For further discussion on this topic, see § I.6 of [3].

1 Sheaves and toposes

We will present an overview of sheaves and their properties, following §§ II and III of [4]. We begin with the simpler case of sheaves on topological spaces in § 1.1, and then consider sheaves on sites in § 1.2.

1.1 Sheaves on topological spaces

A sheaf is an organised collection of sets 'defined locally' on a topological space. For example, if X is a topological space, then the continuous real-valued functions on X will form a sheaf. For each open set $U \subseteq X$, this sheaf records the set of continuous real-valued functions defined on U. The sheaf also tracks some of the topological structure of X. Namely, if $V \subseteq U$, then any continuous map $U \to \mathbb{R}$ may be restricted to form a continuous map $V \to \mathbb{R}$, and this restriction map is part of the data of the sheaf.

Suppose that $f_i : U_i \to \mathbb{R}$ is a family of continuous real-valued functions such that if $x \in U_i \cap U_j$, then $f_i(x) = f_j(x)$. The gluing lemma shows that the f_i may be glued together to form a unique continuous real-valued function $f : \bigcup U_i \to \mathbb{R}$ that agrees with the f_i on their domains. All sheaves are required to satisfy a similar condition called the *gluing property*, or sometimes the *sheaf axiom*, which encapsulates the idea that values in sheaves are given purely by their local data.

Definition 1.1. Let X be a topological space. A *sheaf (of sets)* on X is an assignment of a set $F(U)$ to each open set $U \subseteq X$, together with *restriction maps* $\mathrm{res}^U_V : F(U) \to F(V)$ for each inclusion of open sets $V \subseteq U$, such that the following two conditions hold.[1]

(i) (functoriality) If $W \subseteq V \subseteq U$, then $\mathrm{res}^U_U = 1_U$, and $\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W$.

(ii) (gluing) Let $(U_i)_{i \in I}$ be a collection of open sets that form an open cover of U, and let

$$(x_i \in F(U_i))_{i \in I}$$

be a family such that for each $i, j \in I$,

$$\mathrm{res}^{U_i}_{U_i \cap U_j} x_i = \mathrm{res}^{U_j}_{U_i \cap U_j} x_j$$

Then there is a unique $x \in F(U)$ such that for each $i \in I$,

$$\mathrm{res}^U_{U_i} x = x_i$$

The elements of $F(U)$ are called the *sections* of F at U.

Examples 1.2.

(i) Let X be a topological space. Then we can form the sheaf of continuous real-valued functions on X by

$$F(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\} \qquad \mathrm{res}^U_V(U \xrightarrow{f} \mathbb{R}) = f|_V$$

This forms a sheaf by the discussion above. This example demonstrates how the sheaf axiom can be viewed as a general form of piecewise definition.

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[1] The use of the letter F for a sheaf comes from the French *faisceau*. [4, p. 64]

(ii) [1, p. 62] Let X be an algebraic variety over a field k. Then we can similarly define the sheaf of regular functions on X by

$$F(U) = \{f : U \to k \mid f \text{ regular}\} \qquad \operatorname{res}^U_V(U \xrightarrow{f} k) = f|_V$$

We can similarly define the sheaf of differentiable functions on a differentiable manifold, or the sheaf of holomorphic functions on a complex manifold.

(iii) Let $* = \{\bullet\}$ be the one-point space. A sheaf F on $*$ consists of the data $F(\{\bullet\}), F(\varnothing), \operatorname{res}^{\{\bullet\}}_{\varnothing}$. As the empty family covers the open set $\varnothing$, the set $F(\varnothing)$ must contain exactly one section. This completely determines the restriction map $\operatorname{res}^{\{\bullet\}}_{\varnothing}$. Therefore, a sheaf on $*$ is determined up to isomorphism by the choice of set $F(\{\bullet\})$.

Remarks 1.3.

(i) The data of a sheaf can be described as a functor $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$, where $\mathcal{O}(X)$ is the poset of open sets of X ordered by inclusion, and **Set** is the category of sets. Such a functor is called a *presheaf* on $\mathcal{O}(X)$, so a sheaf on X is precisely a presheaf on $\mathcal{O}(X)$ satisfying the gluing condition (ii) in definition 1.1.

(ii) We can also phrase the gluing condition in more categorical language. Suppose that $(U_i)_{i\in I}$ is an open cover of U. We can define a map

$$e : F(U) \to \prod_{i\in I} F(U_i)$$

by mapping each section x to its restrictions $(\operatorname{res}^U_{U_i} x)_{i\in I}$. We also define maps

$$p, q : \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

by

$$p((x_i)_{i\in I})_{i,j} = \operatorname{res}^{U_i}_{U_i\cap U_j} x_i \qquad q((x_i)_{i\in I})_{i,j} = \operatorname{res}^{U_j}_{U_i\cap U_j} x_j$$

The gluing condition holds precisely when e is an equaliser of p and q for all such open covers $(U_i)_{i\in I}$ of open sets U.

$$\begin{tikzcd} F(U) \arrow[r, "e"] & \prod_{i\in I} F(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j\in I} F(U_i \cap U_j) \end{tikzcd}$$

(iii) In the definition of a sheaf, we required that X be a topological space. The only topological data about X that the definition needed was inclusions and open covers. Motivated by this, we will define a *site* to be a generalisation of a topological space that contains exactly the data required to construct sheaves.

1.2 Sites

Let $\mathcal{C}$ be an arbitrary small category. This will play the role of the poset of opens in a topological space X. An inclusion $V \subseteq U$ will correspond to a morphism $V \to U$ in $\mathcal{C}$. To generalise the notion of cover, we make the following definition.

Definition 1.4. Let A be an object of $\mathcal{C}$. A *sieve* on A is a collection S of morphisms in $\mathcal{C}$ with codomain A that is stable under composition: if $f \in S$ and g is any morphism in $\mathcal{C}$ with $\operatorname{cod} g = \operatorname{dom} f$, then $f \circ g \in S$.

Example 1.5. Let $\mathcal{C} = \mathcal{O}(X)$ be the poset of open sets in a topological space X. Let $(U_i)_{i \in I}$ be a family of open subsets of $U \subseteq X$. Then the family

$$S = \{(1_U)|_V : V \to U \mid \exists i \in I.\, V \subseteq U_i\}$$

forms a sieve on U. It is easy to see that every sieve on U is of this form. Given a sieve S on U, we say that S *covers* U if

$$\bigcup_{f \in S} \operatorname{dom} f = U$$

The sieves that cover U are precisely those that come from open covers of U in the topological sense. We will axiomatise this idea of a covering sieve for arbitrary categories $\mathcal{C}$, giving the notion of a *Grothendieck coverage*.

Definition 1.6. Let $\mathcal{C}$ be a small category. A *Grothendieck coverage*[2] J on $\mathcal{C}$ assigns to each object A of $\mathcal{C}$ a collection of sieves $J(A)$ called the *covering sieves* of A, satisfying the following conditions.

(i) The maximal sieve $\{f \mid \operatorname{cod} f = A\}$ is a covering sieve; that is, $\{f \mid \operatorname{cod} f = A\} \in J(A)$.

(ii) (stability) If S covers A, then for any $h : B \to A$, the sieve

$$h^\star(S) = \{g \mid \operatorname{cod} g = B \wedge g \circ h \in S\}$$

covers B.

(iii) (transitivity) Let S cover A and R be a sieve on A. Suppose that for all $h \in S$, the sieve $h^\star(R)$ is a covering sieve. Then R is a covering sieve.

Remark 1.7. Axioms (i)–(iii) of Grothendieck coverages correspond to familiar facts about open covers in topological spaces.

(i) The family $\{U\}$ covers U.

(ii) If $V \subseteq U$ and the family $(U_i)_{i \in I}$ covers U, then the family $(U_i \cap V)_{i \in I}$ covers V.

(iii) Let $(U_i)_{i \in I}$ cover U, and let $(V_j)_{j \in J}$ be a collection of open sets contained in U. Suppose that for all $i \in I$, the collection $(V_j \cap U_i)_{j \in J}$ is a cover of U_i. Then $(V_j)_{j \in J}$ covers U.

We can now define sites, which will serve as generalisations of topological spaces that are suitable for building sheaves.

Definition 1.8. A *site* is a pair $(\mathcal{C}, J)$, where $\mathcal{C}$ is a small category and J is a Grothendieck coverage on $\mathcal{C}$. A *sheaf* on a site $(\mathcal{C}, J)$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for every J-covering sieve S of some object A, and for every family of sections

$$(x_f \in F(\operatorname{dom} f))_{f \in S}$$

that is compatible in the sense that

$$F(g)(x_f) = x_{f \circ g}$$

whenever this is defined, there is a unique section $x \in F(A)$ such that

$$F(f)(x) = x_f$$

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[2] This name is taken from [2, p. 540]. Some sources, such as [4, p. 110], call this a *Grothendieck topology*.

for each $f \in S$. Equivalently, a sheaf is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for every J-covering sieve S of some object A, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "e"] & \prod_{f\in S} F(\operatorname{dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{\substack{f,g \\ f\in S, \operatorname{dom} f = \operatorname{cod} g}} F(\operatorname{dom} g) \end{tikzcd}$$

is an equaliser, where

$$e(x)_f = F(f)(x) \qquad p(x)_{f,g} = x_{f\circ g} \qquad q(x)_{f,g} = F(g)(x_f)$$

Remarks 1.9.

(i) If X is a topological space, then $(\mathcal{O}(X), J)$ is a site, where J is the Grothendieck coverage such that

$$S \text{ is a } J\text{-covering sieve of } U \leftrightarrow \bigcup_{f\in S} \operatorname{dom} f = U$$

Sheaves on the site $(\mathcal{O}(X), J)$ in the sense of definition 1.8 are precisely sheaves on X in the sense of definition 1.1.

(ii) Generalising remark 1.3 (i), a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$, and a sheaf on $(\mathcal{C}, J)$ is precisely a presheaf on $\mathcal{C}$ that satisfies the gluing property. The presheaves on $\mathcal{C}$ form a category $\mathbf{Psh}(\mathcal{C}) = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, where the morphisms are the natural transformations. We can therefore define the *category of sheaves* on a site $(\mathcal{C}, J)$ to be the full subcategory $\mathbf{Sh}(\mathcal{C}, J)$ of $\mathbf{Psh}(\mathcal{C})$ on those presheaves that satisfy the gluing condition.

(iii) Let $(\mathcal{C}, J)$ be a site. We have a forgetful functor from sheaves on $(\mathcal{C}, J)$ to presheaves on $\mathcal{C}$, called the *inclusion functor*.

$$\mathbf{I} : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Psh}(\mathcal{C})$$

There is also a functor in the other direction, called the *associated sheaf functor*.

$$\mathbf{A} : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$$

The associated sheaf functor is the left adjoint to the inclusion functor; in the language of [3, p. 91], the full subcategory $\mathbf{Sh}(\mathcal{C}, J)$ of $\mathbf{Psh}(\mathcal{C})$ is *reflective*. Furthermore, the left adjoint $\mathbf{A}$ preserves finite limits. For a proof of this fact, see § III.5 of [4]. Note that as $\mathbf{I}$ is full and faithful, the triangle

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "\mathbf{I}"] \arrow[dr, "1"'] & \mathbf{Psh}(\mathcal{C}) \arrow[d, "\mathbf{A}"] \\ & \mathbf{Sh}(\mathcal{C}, J) \end{tikzcd}$$

commutes up to natural isomorphism by [3, p. 90].

1.3 Grothendieck toposes

Definition 1.10. A category $\mathcal{E}$ is called a *Grothendieck topos* if it is equivalent to the category of sheaves on some site.

Grothendieck toposes will be the main objects under consideration in this essay. Among other things, they act as semantic environments for constructive mathematics, with a particular ‘geometric’ flavour. As outlined in the introduction, they have many of the familiar properties of $\mathbf{Set}$. We will begin their study by discussing some basic examples and elementary properties.

Examples 1.11.

- (i) The category of sheaves on a topological space is a Grothendieck topos. In particular, **Set** is a Grothendieck topos because **Set** $\simeq$ **Sh**$(*)$ by example 1.2 (iii).
- (ii) Let $\mathcal{C}$ be a small category, and endow it with the *trivial coverage* J containing only the sieves that are maximal with respect to inclusion; that is, those sieves that include an identity morphism. With respect to this coverage, the sheaf axiom trivially holds for every presheaf on $\mathcal{C}$, so **Psh**$(\mathcal{C}) \simeq$ **Sh**$(\mathcal{C}, J)$. This demonstrates that every presheaf category is a Grothendieck topos. This is another way to see that **Set** is a Grothendieck topos, because **Set** $\simeq$ **Psh**$(\mathbf{1})$, where $\mathbf{1}$ is the terminal category.
- (iii) [4, p. 190] For any fixed set A, the slice category **Set**$/A$ is a Grothendieck topos. To show this, we define $\operatorname{Discr}(A)$ to be the discrete category with objects indexed by elements of A. Then we have an equivalence
$$\mathbf{Set}/A \simeq \mathbf{Psh}(\operatorname{Discr}(A))$$
This equivalence is given in the forward direction by mapping each object $f : B \to A$ to the presheaf $F : \operatorname{Discr}(A)^{\mathrm{op}} \to \mathbf{Set}$ given by
$$F(a) = f^{-1}(a)$$
and in the backward direction by mapping a presheaf $F : \operatorname{Discr}(A)^{\mathrm{op}} \to \mathbf{Set}$ to the natural map
$$\left(\coprod_{a\in A} F(a)\right) \to A$$
In general, slice categories of Grothendieck toposes are Grothendieck toposes; this is a variant of the *fundamental theorem of topos theory*. [2, p. 559] We will not make use of this result.

Remark 1.12.

- (i) Suppose that $D : \mathcal{J} \to \mathbf{Sh}(\mathcal{C}, J)$ is a diagram of sheaves of shape $\mathcal{J}$. It is a general fact that presheaf categories **Psh**$(\mathcal{C})$ have all small limits and colimits and they are computed pointwise, so we can easily compute the limit L of the composite diagram $\mathbf{I}D : \mathcal{J} \to \mathbf{Psh}(\mathcal{C})$. We will show that this limit is a sheaf, then it follows that it is also the apex of a limit cone in **Sh**$(\mathcal{C}, J)$. THis will show that **Sh**$(\mathcal{C}, J)$ has all small limits. We will verify the equaliser condition from definition 1.8. Consider
$$\begin{tikzcd} L(A) \arrow[r, "e"] & \prod_{f\in S} L(\operatorname{dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{\substack{f,g \\ f\in S, \operatorname{dom} f = \operatorname{cod} g}} L(\operatorname{dom} g) \end{tikzcd}$$
As limits in **Psh**$(\mathcal{C})$ are computed pointwise and limits commute with products, this diagram is equivalent to
$$\begin{tikzcd} (\operatorname{Lim} \mathbf{I}D)(A) \arrow[r, "e"] & \operatorname{Lim}\left(\prod_{f\in S} \mathbf{I}D\right)(\operatorname{dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \operatorname{Lim}\left(\prod_{\substack{f,g \\ f\in S, \operatorname{dom} f = \operatorname{cod} g}} \mathbf{I}D\right)(\operatorname{dom} g) \end{tikzcd}$$
But then, as limits commute with equalisers, this is an equaliser diagram since all diagrams of the following form are (pointwise) equalisers.
$$\begin{tikzcd} (\mathbf{I}D)(A) \arrow[r, "e"] & \left(\prod_{f\in S} \mathbf{I}D\right)(\operatorname{dom} f) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \left(\prod_{\substack{f,g \\ f\in S, \operatorname{dom} f = \operatorname{cod} g}} \mathbf{I}D\right)(\operatorname{dom} g) \end{tikzcd}$$

Hence $\mathbf{Sh}(\mathcal{C}, J)$ has all small limits, and they are computed as the limit in $\mathbf{Psh}(\mathcal{C})$:

$$\mathbf{I}\,\mathrm{Lim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J)\right) = \mathrm{Lim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathbf{I}} \mathbf{Psh}(\mathcal{C})\right)$$

As a left adjoint, the associated sheaf functor $\mathbf{A} : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ from remark 1.9 (iii) preserves all colimits that exist in its domain. Now, if $D : \mathcal{J} \to \mathbf{Sh}(\mathcal{C}, J)$, we have

$$\mathrm{Colim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathbf{I}} \mathbf{Psh}(\mathcal{C}) \xrightarrow{\mathbf{A}} \mathbf{Sh}(\mathcal{C}, J)\right) = \mathbf{A}\,\mathrm{Colim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathbf{I}} \mathbf{Psh}(\mathcal{C})\right)$$

But $\mathbf{AI}$ is naturally isomorphic to the identity functor as $\mathbf{A} \dashv \mathbf{I}$ is a reflection, so colimit cocones under $\mathbf{AI}D$ correspond to colimit cocones under D. Hence, $\mathbf{Sh}(\mathcal{C}, J)$ has all small colimits, and they are computed from the colimit in $\mathbf{Psh}(\mathcal{C})$ by taking its associated sheaf.

$$\mathrm{Colim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J)\right) = \mathbf{A}\,\mathrm{Colim}\left(\mathcal{J} \xrightarrow{D} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{\mathbf{I}} \mathbf{Psh}(\mathcal{C})\right)$$

(ii) [3, p. 61] There is a full and faithful functor

$$\mathbf{Y} : \mathcal{C} \to \mathbf{Psh}(\mathcal{C})$$

which maps each object A to its contravariant hom-functor $\mathrm{Hom}_{\mathcal{C}}(-, A)$, and maps each morphism $f : A \to B$ to the natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, B)$$

given by composing with f on the left. This functor is called the *Yoneda embedding*. In this language, the Yoneda lemma for a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ can be written

$$F(A) \cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{Y}A, F)$$

The most common way to pass from a site to its Grothendieck topos is to make use of the composite

$$\mathcal{C} \xrightarrow{\mathbf{Y}} \mathbf{Psh}(\mathcal{C}) \xrightarrow{\mathbf{A}} \mathbf{Sh}(\mathcal{C}, J)$$

Presheaves that are naturally isomorphic to $\mathbf{Y}A$ for some A are called *representable*. If the coverage J is such that all representable presheaves are sheaves, there is no need to take associated sheaves, and so this composite has a simpler form. This is a property of certain coverages that will become useful in § 4.

1.4 Geometric morphisms

Let $f : X \to Y$ be a continuous map of topological spaces. There is a *direct image* functor $f_\star : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ given by

$$(f_\star F)(V) = F(f^{-1}(V))$$

It is easy to check that this is a sheaf. One can show that this has a left adjoint, called the *inverse image* functor $f^\star : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.

$$f^\star \dashv f_\star$$

Moreover, $f^\star$ preserves finite limits. The construction of the inverse image map is detailed in [4, p. 99].

Under certain mild conditions, every adjoint pair whose left adjoint preserves finite limits arises in this way, up to natural isomorphism. It suffices to assume that X and Y are *sober*: every irreducible closed set is the closure of a unique point. This is satisfied if, for example, X and Y are Hausdorff. A proof can be found in [2, p. 515], using the equivalence between spatial locales and sober spaces from [2, p. 491].

Definition 1.13. A *geometric morphism* $f : \mathcal{E} \to \mathcal{F}$ between Grothendieck toposes is an adjoint pair

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend right, "f_\star"'] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend right, "f^\star"'] \end{tikzcd}$$

where the left adjoint $f^\star$ preserves finite limits.

Recall from [3, p. 272] that a *2-category* $\mathfrak{C}$ is a category $\mathcal{C}$ together with a collection of *2-cells* $\alpha : f \Rightarrow g$ where $f, g : A \to B$ are parallel morphisms (*1-cells*) of $\mathcal{C}$, subject to certain laws concerning the two notions of composition which we will not discuss. For example, $\mathfrak{Cat}$ is the 2-category where the objects are the categories, the 1-cells are the functors, and the 2-cells are the natural transformations.[3] We write $\mathfrak{C}(A, B)$ for the (1-)category whose objects are the morphisms $A \to B$ in $\mathfrak{C}$ and whose morphisms $f \to g$ are the 2-cells $f \Rightarrow g$ in $\mathfrak{C}$, so $\mathfrak{Cat}(F, G)$ is the category $[F, G]$.

Definition 1.14. We define $\mathfrak{Top}$ to be the 2-category where the objects are the Grothendieck toposes, the 1-cells $\mathcal{E} \to \mathcal{F}$ are the geometric morphisms $f : \mathcal{E} \to \mathcal{F}$, and the 2-cells $f \Rightarrow g$ are the natural transformations $\alpha : f_\star \to g_\star$ (or equivalently by adjointness, natural transformations $\beta : g^\star \to f^\star$).

Example 1.15. The inclusion functor and associated sheaf functor from remark 1.9 (iii) form a geometric morphism

$$(\mathbf{A} \dashv \mathbf{I}) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Psh}(\mathcal{C})$$

Remark 1.16. It can be shown that the conditions of the special adjoint functor theorem [3, p. 128] are always satisfied for functors between Grothendieck toposes. Therefore, such a functor is a left (resp. right) adjoint if and only if it preserves all small colimits (resp. limits). In particular, a functor between Grothendieck toposes is the inverse image part of a geometric morphism if and only if it preserves finite limits and small colimits. [2, p. 554]

The two parts of a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ preserve different parts of the structure of $\mathcal{E}$ and $\mathcal{F}$. The direct image part $f_\star$ preserves the geometric structure of $\mathcal{E}$; we have seen this in the special case of sober topological spaces, as each such $f_\star$ comes from a unique continuous map $X \to Y$. We will see that the inverse image part $f^\star$ preserves the algebraic structure of $\mathcal{F}$. More concretely, $f^\star$ will induce a functor that maps models of a given theory in $\mathcal{F}$ to models of the same theory in $\mathcal{E}$.

1.5 Exponential objects

The objects of Grothendieck toposes behave like the types in a type theory, and can be combined using many of the common type-building rules. We have already shown that Grothendieck toposes admit all set-indexed product and coproduct types, as well as pullbacks, pushouts, equalisers, and coequalisers. In the remainder of this section, we will explore more of the type-building rules, which will allow us to interpret a suitable form of first-order logic in any Grothendieck topos. We begin by showing that Grothendieck toposes are cartesian closed, using the construction from [4, p. 136].

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[3] We are not concerned here with size issues.

First, we consider presheaf categories $\mathbf{Psh}(\mathcal{C})$. Let P, Q be presheaves on $\mathcal{C}$. If the exponential Q^P exists, for each object A of $\mathcal{C}$ it must satisfy

$$\begin{aligned} Q^P(A) &\cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{Y}A, Q^P) && \text{by the Yoneda lemma} \\ &\cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{Y}A \times P, Q) && \text{by definition of the exponential} \end{aligned}$$

We claim that this definition works; that is, the presheaf

$$Q^P = \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{Y}(-) \times P, Q)$$

satisfies the universal property of the exponential. For a morphism $\alpha : R \times P \to Q$, we obtain a morphism $\widetilde{\alpha} : R \to Q^P$ given by

$$(\widetilde{\alpha}_A(x \in RA))_B(B \xrightarrow{f} A, y \in PB) = \alpha_B(Rfx, y) \in QB$$

and for a morphism $\beta : R \to Q^P$, we obtain $\widetilde{\beta} : R \times P \to Q$ by

$$\widetilde{\beta}_A(x \in RA, y \in PA) = (\beta_A x)_A(1_A, y) \in QA$$

It is easy to check that these are inverses and have the required naturality properties, giving the adjunction as required.

Now, consider general sheaf categories $\mathbf{Sh}(\mathcal{C}, J)$. One can show that if F, G are sheaves on $(\mathcal{C}, J)$, then the presheaf exponential G^F as defined above is always a sheaf; the proof is long, but can be found in [4, p. 136]. The presheaf exponential satisfies the universal property of the exponential in $\mathbf{Sh}(\mathcal{C}, J)$: as $\mathbf{I}$ is full and faithful and preserves finite products, we have suitably natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C},J)}(H, G^F) &\cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{I}H, \mathbf{I}(G^F)) \\ &= \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{I}H, \mathbf{I}G^{\mathbf{I}F}) \\ &\cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{I}H \times \mathbf{I}F, \mathbf{I}G) \\ &\cong \mathrm{Hom}_{\mathbf{Psh}(\mathcal{C})}(\mathbf{I}(H \times F), \mathbf{I}G) \\ &\cong \mathrm{Hom}_{\mathbf{Sh}(\mathcal{C},J)}(H \times F, G) \end{aligned}$$

as required.

1.6 Subobjects

Monomorphisms into an object A are preordered by defining that $m \leq n$ if and only if m factors through n.

$$\begin{tikzcd} S \arrow[r, dashed] \arrow[dr, "m"'] & T \arrow[d, "n"] \\ & A \end{tikzcd}$$

The posetal reflection of this preorder is the category of *subobjects* of A, denoted $\mathrm{Sub}(A)$. Recall that a category $\mathcal{E}$ is *well-powered* if all of its subobject posets are small. If $\mathcal{E}$ also has pullbacks, we can turn Sub into a functor $\mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ 'by pullback', mapping a morphism $f : A \to B$ to the function $\mathrm{Sub}(B) \to \mathrm{Sub}(A)$ given by sending a monomorphism $m : S \rightarrowtail B$ to its pullback m' along f.

$$\begin{tikzcd} T \arrow[r] \arrow[d, tail, "m'"'] & S \arrow[d, tail, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

This map $\mathrm{Sub}(f)$ is a functor $\mathrm{Sub}(B) \to \mathrm{Sub}(A)$ as it is order-preserving as a map of posets. As an example, if $f : A \to B$ is a function in **Set**,

$$\mathrm{Sub}(f)(S) = \{x \in A \mid f(x) \in S\} = f^{-1}S$$

We will often refer to $\mathrm{Sub}(f)$ using the notation f^{-1} to reflect its behaviour in **Set**. In addition, we will frequently abuse language by conflating subobjects with their representatives, and representative monomorphisms with their domains.

In **Set**, subobjects of A correspond bijectively to their characteristic functions $A \to 2$. A version of this correspondence holds for every Grothendieck topos.

Definition 1.17. Let $\mathcal{E}$ be a well-powered category with finite limits. A monomorphism

$$\mathrm{true} : 1 \rightarrowtail \Omega$$

is called a *subobject classifier* of $\mathcal{E}$ if for every monomorphism $m : S \rightarrowtail A$, there is a unique *characteristic function* $\chi : A \to \Omega$ with a pullback square

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\mathrm{true}"] \\
A \arrow[r, "\chi"'] & \Omega
\end{tikzcd}$$

If Ω is a subobject classifier of $\mathcal{E}$, we have a natural isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(-) \cong \mathrm{Hom}_{\mathcal{E}}(-, \Omega)$$

Therefore, Ω is a representing object for subobjects.

Examples 1.18.

(i) In **Set**, the map $\mathrm{true} : 1 \to 2$ given by $\mathrm{true}(0) = 1$ is a subobject classifier. The characteristic function of a subset $S \subseteq A$ is

$$\chi(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases}$$

(ii) [4, p. 98] Any subobject S of a sheaf F can be uniquely represented as a *subsheaf* of F; that is, as a sheaf that is a subset of F on every section, and where the map $S \to F$ is the sectionwise inclusion. Note that the order on subobjects corresponds exactly to the order on representing subsheaves.

We can now define the subobject classifier of $\mathbf{Sh}(X)$: it is given by

$$\Omega(U) = \{V \subseteq U \text{ open}\}$$

The characteristic natural transformation $\chi : F \to \Omega$ for a subsheaf S of F is given by

$$\chi_U(x) = \bigcup \{W \subseteq U \text{ open} \mid \mathrm{res}^U_W\, x \in SW\}$$

In particular, the subobjects of 1 in $\mathbf{Sh}(X)$ correspond to the open sets of X. Analogously to the role of open sets in Heyting semantics, subobjects of 1 will be the truth values of sentences in the logic that we will define.

(iii) [4, p. 140] For a general site $(\mathcal{C}, J)$, we can define $\Omega(A)$ to be the set of those sieves S on A such that for all morphisms f with codomain A, if the sieve $f^\star S$ covers $\operatorname{dom} f$, then $f \in S$. If $S \subseteq F$ is a subsheaf, we define its characteristic natural transformation $\chi : F \to \Omega$ by

$$\chi_A(x) = \{f : B \to A \mid Ffx \in SB\}$$

It is easy to check that these constructions agree with those in (ii) in the case where $(\mathcal{C}, J)$ arises from a topological space.

Remarks 1.19.

(i) Every object in a Grothendieck topos has a *power object* $PA = \Omega^A$. For each B, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(B \times A, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(A, PB)$$

which is natural in A. In particular, subobjects of A correspond to morphisms $1 \to PA$ by

$$\mathrm{Sub}_{\mathcal{E}}(A) \cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(A \times 1, \Omega) \cong \mathrm{Hom}_{\mathcal{E}}(1, PA)$$

In general, morphisms $1 \to B$ are called the *global elements* of B, so we may equivalently say that subobjects of a type correspond to global elements of its power object.

(ii) [4, p. 145] Let $(S_i)_{i \in I}$ be a small family of subobjects of an object F in $\mathbf{Sh}(\mathcal{C}, J)$. As in example 1.18 (ii), without loss of generality we may assume that the S_i are all subsheaves of F. This shows that the $(S_i)_{i \in I}$ have a meet in $\mathrm{Sub}(F)$, which is given by their sectionwise intersection.

$$\left(\bigwedge_{i \in I} S_i \right)(B) = \bigcap_{i \in I} S_i(B)$$

We can alternatively describe this meet as the wide pullback of the S_i; that is, the limit of the diagram consisting of all of the subobjects. This is an easy way to show that this meet is a sheaf, because wide pullbacks commute with equalisers. Therefore, $\mathrm{Sub}(F)$ is a complete lattice, because

$$\bigvee_{i \in I} S_i = \bigwedge \{T \text{ a subsheaf of } F \mid \forall i \in I.\, S_i \text{ is a subsheaf of } T\}$$

The meet and join operations satisfy an infinitary distributive law, namely

$$S \wedge \bigvee_{i \in I} T_i = \bigvee_{i \in I} (S \wedge T_i)$$

This is precisely the statement that the functor $S \wedge (-)$ preserves all colimits in $\mathrm{Sub}(F)$, which implies that $S \wedge (-)$ has a right adjoint by the adjoint functor theorem for posets. This makes $\mathrm{Sub}(F)$ into a complete Heyting algebra. We will often refer to the meet and join operations as intersection and union due to their forms in **Set**.

(iii) [4, p. 147] Let $\alpha : F \to G$ be a morphism in a Grothendieck topos $\mathcal{E}$. Then the pullback functor $\alpha^{-1} : \mathrm{Sub}(G) \to \mathrm{Sub}(F)$ has both a left adjoint $\exists_\alpha$ and a right adjoint $\forall_\alpha$.

In **Set**, the adjoints take the following simple form. If $\alpha : F \to G$ is a function of sets and $S \subseteq F$, then

$$\exists_\alpha(S) = \{\alpha(x) \mid x \in S\} \qquad \forall_\alpha(S) = \{y \mid \alpha^{-1}(y) \subseteq S\}$$

Alternatively,

$$
\begin{aligned}
y \in \exists_\alpha(S) &\leftrightarrow (\exists x \in S.\, \alpha(x) = y) \\
y \in \forall_\alpha(S) &\leftrightarrow (\forall x \in F.\, \alpha(x) = y \to x \in S)
\end{aligned}
$$

The left adjoint is the image under α; the right adjoint is sometimes called the *kernel image* under α. We can use these adjoints to interpret existential and universal quantification, justifying their notation. Let $\varphi(x, y)$ be a property of pairs $(x, y) \in A \times B$, and let $S \subseteq A \times B$ be the set of pairs that satisfy φ. Then, if $\alpha = \pi_1 : A \times B \to A$ is the first projection,

$$
\begin{aligned}
a \in \exists_{\pi_1}(S) &\leftrightarrow \exists b \in B.\, \varphi(a, b) \\
a \in \forall_{\pi_1}(S) &\leftrightarrow \forall b \in B.\, \varphi(a, b)
\end{aligned}
$$

Therefore $\exists_{\pi_1}(S)$ is the set of objects $a \in A$ that satisfy $\exists b.\, \varphi(a, b)$, and similarly for $\forall_{\pi_1}(S)$.

We can give the adjoints an explicit description in an arbitrary Grothendieck topos $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$. If $\alpha : F \to G$ is a morphism in $\mathbf{Sh}(\mathcal{C}, J)$ and S is a subsheaf of F, then $\exists_\alpha(S)$ and $\forall_\alpha(S)$ are the subsheaves of F given by

$$
\begin{aligned}
y \in \exists_\alpha(S)(A) &\leftrightarrow \{f : B \to A \mid \exists x \in S(B).\, \alpha_B(x) = G(f)(y)\} \text{ covers } A \\
y \in \forall_\alpha(S)(A) &\leftrightarrow (\forall f : B \to A.\, \forall x \in F(B).\, \alpha_B(x) = G(f)(y) \to x \in S(B))
\end{aligned}
$$

where A is an object of $\mathcal{C}$ and $y \in G(A)$. It may be shown that these are sheaves and give the required adjunctions.

1.7 Image factorisations

Every map $f : A \to B$ in $\mathcal{E}$ has an *image factorisation*

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[dr, "f"'] & M \arrow[d, tail, "m"] \\
& B
\end{tikzcd}$$

where e is an epimorphism, and m is the smallest subobject of B through which f factors. The subobject given by m is called the *image* of f, and is written $\operatorname{im} f$. Moreover, this factorisation is unique up to isomorphism. It can be obtained, for example, by considering the equaliser of the cokernel pair of f, or dually, the coequaliser of the kernel pair of f. [4, p. 184]

We can also define the left adjoint $\exists_f$ to the pullback functor f^{-1} in terms of images. [4, p. 187] If $m : S \rightarrowtail A$ is a subobject, then $\exists_f(S)$ is the image of fm. To show this, we verify that this construction satisfies the universal property of the left adjoint to f^{-1}; that is, for any subobject $T \rightarrowtail B$,

$$\operatorname{im}(fm) \le T \leftrightarrow S \le f^{-1}T$$

Suppose we have a morphism of subobjects $p : \operatorname{im}(fm) \to T$. Then we obtain a unique morphism $q : S \to f^{-1}T$ making the following diagram commute:

$$\begin{tikzcd}
S \arrow[rrr, two heads, "e"] \arrow[dr, dashed, "q"] \arrow[ddr, tail, bend right, "m"'] & & & \operatorname{im}(fm) \arrow[dl, dashed, "p"'] \arrow[ddl, tail, bend left] \\
& f^{-1}T \arrow[r] \arrow[d, tail] & T \arrow[d, tail] & \\
& A \arrow[r, "f"'] & B &
\end{tikzcd}$$

where the square is a pullback which gives rise to q by its universal property. Conversely, if we have such a q, the diagram gives a factorisation of fm through T. But then $\mathrm{im}(fm)$ must also factorise through T, giving a unique p.

Remark 1.20.

(i) If $f : A \to B$ is both monic and epic, then $f1_A$ and 1_Bf are both image factorisations, so are isomorphic by uniqueness.

$$\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[d, equal] & B \arrow[r, tail, "1_B"] \arrow[d, "g"'] & B \arrow[d, equal] \\
A \arrow[r, two heads, "1_A"'] & A \arrow[r, tail, "f"'] & B
\end{tikzcd}$$

Therefore, f must be an isomorphism. This shows that all Grothendieck toposes are *balanced* categories.

(ii) Given a small family $(S_i)_{i\in I}$ of subobjects of A, we can form their coproduct $\coprod_{i\in I} S_i$. We can define a map from this coproduct into A by setting its composite with the ith coprojection to be the inclusion of S_i into A.

$$\begin{tikzcd}
S_i \arrow[r, "\nu_i"] \arrow[dr, tail] & \coprod_{i\in I} S_i \arrow[d, "g"] \\
 & A
\end{tikzcd}$$

Then, the image of g is precisely the union of the S_i.

(iii) It is easy to verify that inverse image parts of geometric morphisms preserve image factorisations and finite intersections, and (ii) shows that they also preserve small unions. In general, arbitrary small intersections are not preserved.

1.8 Giraud's theorem

There is a famous theorem due to Giraud that characterises Grothendieck toposes without making reference to sheaves or sites. It is included for completeness, but we will not make use of it in this essay.

Definition 1.21. An *equivalence relation* on an object A of $\mathcal{E}$ is a subobject $(\partial_0, \partial_1) : R \rightarrowtail A \times A$ such that

(i) (reflexivity) the diagonal $\Delta : A \to A \times A$ factors through (∂_0, ∂_1);

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[dr, "\Delta"'] & R \arrow[d, tail, "{(\partial_0,\partial_1)}"] \\
 & A \times A
\end{tikzcd}$$

(ii) (symmetry) (∂_1, ∂_0) factors through (∂_0, ∂_1);

$$\begin{tikzcd}
R \arrow[r, dashed] \arrow[dr, "{(\partial_1,\partial_0)}"'] & R \arrow[d, tail, "{(\partial_0,\partial_1)}"] \\
 & A \times A
\end{tikzcd}$$

(iii) in the pullback

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "g"'] & R \arrow[d, "\partial_0"] \\
R \arrow[r, "\partial_1"'] & A
\end{tikzcd}$$

the morphism $(\partial_0 g, \partial_1 f) : B \to A \times A$ factors through R.

$$\begin{tikzcd}
B \arrow[r, dashed] \arrow[dr, "(\partial_0 g{,}\partial_1 f)"'] & R \arrow[d, tail, "(\partial_0{,}\partial_1)"] \\
& A \times A
\end{tikzcd}$$

Theorem 1.22 (Giraud). [4, p. 577] A category $\mathcal{E}$ is a Grothendieck topos if and only if

(i) $\mathcal{E}$ is locally small;

(ii) $\mathcal{E}$ has a generating set (in the sense of [4, p. 139]);

(iii) $\mathcal{E}$ has all finite limits;

(iv) $\mathcal{E}$ has all small coproducts, their coprojections have empty intersection as subobjects, and they are stable under pullback;

(v) every epimorphism in $\mathcal{E}$ is a coequaliser;

(vi) every equivalence relation $R \rightrightarrows A$ in $\mathcal{E}$ is a kernel pair and has a coequaliser; and

(vii) if

$$\begin{tikzcd}
R \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & A \arrow[r, "q"] & Q
\end{tikzcd}$$

has the property that q is the coequaliser of ∂_0 and ∂_1, and (∂_0, ∂_1) are the kernel pair of q, then this property is preserved under pullback along any map $Q' \to Q$.

2 Toposes as semantic environments

The type-building rules in § 1 allow us to interpret a suitable form of first-order logic in any Grothendieck topos. We will only be able to interpret suitably 'geometric' formulas; in particular, we cannot interpret concepts like classical negation, since the subobject lattices in Grothendieck toposes need not have complements. This section will follow [2, pp. 807–828].

2.1 Signatures

Definition 2.1. A *signature* Σ consists of

(i) a set of symbols called *sorts*;

(ii) a set of *function symbols*, each with a list of sorts called its *domain* and a sort called its *codomain*; and

(iii) a set of *relation symbols*, each with a list of sorts called its *type*.

We write lists with square brackets, as in $[A_1, \dots, A_n]$, and use an arrow over the name of a variable to signify that it is a list, as in $\vec{A}$. If f is a function symbol with domain $[A_1, \dots, A_n]$ and codomain B, we may write

$$f : [A_1, \dots, A_n] \to B$$

and if R is a relation symbol of type $[A_1, \dots, A_n]$, we write

$$R \rightarrowtail [A_1, \dots, A_n]$$

This notation is chosen to reflect the intended interpretation of function and relation symbols in a Grothendieck topos, which we will now define.

Definition 2.2. Let $\mathcal{E}$ be a Grothendieck topos and let Σ be a signature. A Σ-*structure* M in $\mathcal{E}$ consists of

(i) for each sort A of Σ, an object MA of $\mathcal{E}$;

(ii) for each function symbol $f : [A_1, \dots, A_n] \to B$ of Σ, a morphism

$$Mf : MA_1 \times \cdots \times MA_n \to MB$$

in $\mathcal{E}$;

(iii) for each relation symbol $R \rightarrowtail [A_1, \dots, A_n]$ of Σ, a subobject

$$MR \rightarrowtail MA_1 \times \cdots \times MA_n$$

in $\mathcal{E}$.

For brevity, we will sometimes write $M\vec{A}$ for $MA_1 \times \cdots \times MA_n$. We make the collection of Σ-structures in $\mathcal{E}$ into a category by defining the morphisms as follows.

Definition 2.3. Let M, N be Σ-structures in $\mathcal{E}$. A *morphism of* Σ-*structures* $h : M \to N$ consists of a morphism $h_A : MA \to NA$ for each sort A of Σ, such that

(i) for every function symbol $f : \vec{A} \to B$ in Σ, the square

$$\begin{tikzcd}
M\vec{A} \arrow[r, "Mf"] \arrow[d, "h_{\vec{A}}"'] & MB \arrow[d, "h_B"] \\
N\vec{A} \arrow[r, "Nf"'] & NB
\end{tikzcd}$$

commutes; and

(ii) for every relation symbol $R \rightarrowtail \vec{A}$ in Σ, there is a map $MR \to NR$ such that the square

$$\begin{tikzcd}
MR \arrow[r, tail] \arrow[d] & M\vec{A} \arrow[d, "h_{\vec{A}}"] \\
NR \arrow[r, tail] & N\vec{A}
\end{tikzcd}$$

commutes.

The category of Σ-structures in $\mathcal{E}$ will be denoted $\Sigma\text{-Str}(\mathcal{E})$.

Remark 2.4. If $T : \mathcal{F} \to \mathcal{E}$ is a functor between Grothendieck toposes that preserves finite limits, then we have a functor

$$\Sigma\text{-Str}(T) : \Sigma\text{-Str}(\mathcal{F}) \to \Sigma\text{-Str}(\mathcal{E})$$

because T must preserve finite products and monomorphisms. Usually, T will be the inverse image part of a geometric morphism. For brevity, we will often write T for Σ-Str(T).

2.2 Terms

We can easily define terms for our logic, although we need to give some details about their interpretations. As usual, we assume a countably infinite supply of variables of each sort.

Definition 2.5. Let Σ be a signature. A *term* over Σ is defined inductively as follows.

(i) If x is a variable of sort A, then x is a term of sort A.

(ii) If $f : [A_1, \dots, A_n] \to B$ is a function symbol of Σ, and $t_1, \dots, t_n$ are terms of sort $A_1, \dots, A_n$, then $f(t_1, \dots, t_n)$ is a term of sort B.

If t is a term of sort A, we may write $t : A$. The interpretation of a term $t : B$ with variables $x_1 : A_1, \dots, x_n : A_n$ will be a morphism

$$MA_1 \times \cdots \times MA_n \to MB$$

In particular, the interpretation of a closed term of sort B will be a global element of MB. To help keep track of the variables used within a term, we make the following definition.

Definition 2.6. A *context* over Σ is a (finite) list of distinct variables.

$$\vec{x} = [x_1, \dots, x_n]$$

A *term-in-context* $\vec{x} \,.\, t$ is a term t together with a context $\vec{x}$ that contains all of the variables that occur in t.

Note that contexts need not be *minimal*; variables may appear in a context even if they do not appear in the term. We can now define the interpretation of a term-in-context in a structure.

Definition 2.7. Let M be a Σ-structure and let

$$[x_1 : A_1, \dots, x_n : A_n] \,.\, t : B$$

be a Σ-term-in-context. The *interpretation* of $\vec{x} \,.\, t$ in M is the morphism

$$[\![\vec{x} \,.\, t]\!] : M\vec{A} \to MB$$

given inductively as follows.

(i) If $t \equiv x_i$ is a variable, then $[\![\vec{x} \,.\, x_i]\!]$ is the ith projection

$$M\vec{A} \xrightarrow{\pi_i} MA_i$$

(ii) If $t \equiv f(t_1, \dots, t_m)$ is a function application with $f : \vec{B} \to C$, then $[\![\vec{x} \,.\, f(t_1, \dots, t_m)]\!]$ is the composite

$$M\vec{A} \xrightarrow{([\![\vec{x} \,.\, t_1]\!], \dots, [\![\vec{x} \,.\, t_m]\!])} M\vec{B} \xrightarrow{Mf} MC$$

Remark 2.8. Interpretation of terms satisfies various naturality conditions.

(i) Let $\left(\vec{x} : \vec{A}\right) . t$ be a term-in-context of sort C, and let $\vec{y} : \vec{B}$ be a minimal context for t. Then there is a commutative triangle

$$\begin{tikzcd}
M\vec{A} \arrow[r, dashed] \arrow[dr, "{[\![\vec{x} . t]\!]}"'] & M\vec{B} \arrow[d, "{[\![\vec{y} . t]\!]}"] \\
 & MC
\end{tikzcd}$$

where the dashed morphism is a list of projections. This shows that the context in which we interpret a term does not matter.

(ii) Interpretation of terms is natural in the sense that for any homomorphism of Σ-structures $h : M \to N$, all squares of the form

$$\begin{tikzcd}
M\vec{A} \arrow[r, "{[\![\vec{x} . t]\!]_M}"] \arrow[d, "h_{\vec{A}}"'] & MB \arrow[d, "h_B"] \\
N\vec{A} \arrow[r, "{[\![\vec{x} . t]\!]_N}"'] & NB
\end{tikzcd}$$

commute.

(iii) By remark 2.4, any functor $T : \mathcal{F} \to \mathcal{E}$ that preserves finite limits gives rise to a functor

$$\Sigma\text{-Str}(T) : \Sigma\text{-Str}(\mathcal{F}) \to \Sigma\text{-Str}(\mathcal{E})$$

It is easy to see that this functor preserves the interpretation of terms. Indeed, if M is a Σ-structure in $\mathcal{F}$, then for any term-in-context $\vec{x} . t$, we have

$$T\,[\![\vec{x} . t]\!]_M = [\![\vec{x} . t]\!]_{TM}$$

The proof only requires that $f^{\star}$ preserves finite limits.

2.3 Formulas

We will restrict ourselves to a fragment of infinitary first-order logic called *geometric logic*, in which the only permitted symbols are equalities, relation symbols, finitary conjunctions, infinitary disjunctions, and existential quantification. The interpretations of these formulas behave nicely under inverse images of geometric morphisms.

Definition 2.9. The set of *(geometric) formulas* over a signature Σ is defined inductively as follows.

(i) If s, t are Σ-terms, then $(s = t)$ is a geometric formula.

(ii) If $R \rightarrowtail [A_1, \dots, A_m]$ is a relation symbol in Σ and $t_1 : A_1, \dots, t_m : A_m$ are terms, then $R(t_1, \dots, t_m)$ is a geometric formula.

(iii) If $(\varphi_i)_{i<m}$ are geometric formulas and $m \in \omega$, then $\bigwedge_{i<m} \varphi_i$ is a geometric formula.

(iv) If $(\varphi_i)_{i \in I}$ are geometric formulas whose free variables all come from a finite set, then $\bigvee_{i \in I} \varphi_i$ is a geometric formula.

(v) If φ is a geometric formula and x is a variable of sort A, then $\exists x : A . \varphi$ is a geometric formula.

Occasionally, we will write a formula φ with free variables $\vec{x}$ as $\varphi(\vec{x})$ to simplify the syntax for substitution.

We can similarly define a *formula-in-context* $\vec{x}\,.\,\varphi$ to be a formula together with a context that contains all of its free variables. The requirement in (iv) that the free variables all come from a finite set ensures that every geometric formula has a context. Using this, we can define the interpretation of geometric formulas in a structure in a straightforward way.

Definition 2.10. Let $\vec{x}\,.\,\varphi$ be a formula-in-context for a signature Σ, and let M be a structure. The *interpretation* of $\vec{x}\,.\,\varphi$ in M is the subobject

$$[\![\vec{x}\,.\,\varphi]\!] \rightarrowtail M\vec{A}$$

defined inductively as follows.

(i) If $\varphi \equiv (s = t)$, then $[\![\vec{x}\,.\,\varphi]\!]$ is the equaliser of

$$\begin{tikzcd} M\vec{A} \arrow[r, shift left, "{[\![\vec{x}\,.\,s]\!]}"] \arrow[r, shift right, "{[\![\vec{x}\,.\,t]\!]}"'] & MB \end{tikzcd}$$

(ii) If $\varphi \equiv R(t_1, \dots, t_m)$, then $[\![\vec{x}\,.\,\varphi]\!]$ is the upper morphism in the pullback square

$$\begin{tikzcd} {[\![\vec{x}\,.\,\varphi]\!]} \arrow[r, tail] \arrow[d] & M\vec{A} \arrow[d, "{([\![\vec{x}\,.\,t_1]\!],\dots,[\![\vec{x}\,.\,t_m]\!])}"] \\ MR \arrow[r, tail] & M\vec{B} \end{tikzcd}$$

(iii) If $\varphi \equiv \bigwedge_{i<m} \varphi_i$, then $[\![\vec{x}\,.\,\bigwedge_{i<m} \varphi_i]\!]$ is the intersection of the subobjects $[\![\vec{x}\,.\,\varphi_i]\!]$.

(iv) If $\varphi \equiv \bigvee_{i\in I} \varphi_i$, then $[\![\vec{x}\,.\,\bigvee_{i\in I} \varphi_i]\!]$ is the union of the subobjects $[\![\vec{x}\,.\,\varphi_i]\!]$.

(v) If $\varphi \equiv \exists y : B.\,\psi$, where without loss of generality y does not appear in $\vec{x}$, then $[\![\vec{x}\,.\,\exists y.\,\psi]\!]$ is the image of the composite

$$\begin{tikzcd} {[\![[\vec{x}, y]\,.\,\psi]\!]} \arrow[r, tail] & M\vec{A} \times MB \arrow[r, "\pi_1"] & M\vec{A} \end{tikzcd}$$

Remark 2.11.

(i) By the discussion in § 1.7 on representing $\exists_f$ in terms of image factorisations, we obtain an equality of subobjects

$$[\![\vec{x}\,.\,\exists y.\,\psi]\!] = \exists_{\pi_1} [\![[\vec{x}, y]\,.\,\psi]\!]$$

where π_1 is the projection $M\vec{A} \times MB \to M\vec{A}$.

(ii) The interpretation of the empty conjunction $\top$ in context $\vec{x}$ is the identity map $M\vec{A} \rightarrowtail M\vec{A}$. This is the maximal subobject. Similarly, the interpretation of the empty disjunction is the minimal subobject $0 \rightarrowtail M\vec{A}$.

(iii) Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. The functor

$$\Sigma\text{-Str}(f^\star) : \Sigma\text{-Str}(\mathcal{F}) \to \Sigma\text{-Str}(\mathcal{E})$$

also preserves interpretations of formulas, in the sense that if $\vec{x} \,.\, \varphi$ is a formula-in-context, then

$$f^\star \left[\!\left[\vec{x} \,.\, \varphi\right]\!\right]_M = \left[\!\left[\vec{x} \,.\, \varphi\right]\!\right]_{f^\star M}$$

as subobjects of $f^\star(M\vec{A})$. This is easily proven by induction on the length of formulas. In general, such an inverse image functor need not preserve interpretations of non-geometric formulas, such as implication or universal quantification.[4]

(iv) Suppose that φ is a formula with no free variables, so it can be given the empty context $[]$. Then the interpretation of $[] \,.\, \varphi$ in a structure M is a subobject of 1, or equivalently, a global element of Ω. If $\mathcal{E} \simeq \mathbf{Sh}(X)$, then the global elements of Ω correspond to the open sets of X. Therefore, in such a Grothendieck topos, the truth values of geometric sentences correspond bijectively to the open sets. For instance, in **Set**, there are exactly two geometric truth values, which correspond to classical truth and falsity.

2.4 Theories

Classically, a theory is a set of sentences that models are required to satisfy. Since our geometric formulas may take truth values other than 'true' and 'false', we will need a non-classical notion of satisfaction. Despite this, comparisons between truth values (or, more generally, subobjects) are always either true or false, since they form an external partial order. Motivated by this observation, one natural definition to make is the following.

Definition 2.12. A *(geometric) sequent* over Σ is a formal expression

$$\varphi \vdash_{\vec{x}} \psi$$

where $\vec{x} : \vec{A}$ is a context and $\vec{x} \,.\, \varphi, \vec{x} \,.\, \psi$ are geometric formulas-in-context for Σ. A sequent $\varphi \vdash_{\vec{x}} \psi$ is said to be *satisfied* by a Σ-structure M if

$$\left[\!\left[\vec{x} \,.\, \varphi\right]\!\right]_M \leq \left[\!\left[\vec{x} \,.\, \psi\right]\!\right]_M$$

as subobjects of $M\vec{A}$.

Satisfaction of geometric sequents is clearly preserved by inverse image parts of geometric morphisms.

Remark 2.13. In classical logic,

$$\varphi \vdash_{\vec{x}} \psi \quad \text{corresponds to} \quad \forall \vec{x} : \vec{A}.\, \varphi(\vec{x}) \to \psi(\vec{x})$$

in the sense that the sequent $\varphi \vdash_{\vec{x}} \psi$ is satisfied in a Σ-structure M in **Set** if and only if the sentence $\forall \vec{x} : \vec{A}.\, \varphi(\vec{x}) \to \psi(\vec{x})$ classically holds in M. It turns out that many theories are classically equivalent to one using only sentences of the form $\forall \vec{x}.\, \varphi \to \psi$, where φ, ψ are geometric. Therefore, such a

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[4] A sufficient condition for such an inverse image functor to preserve interpretations of implication, universal quantification, and constructive negation, is that $f : \mathcal{E} \to \mathcal{F}$ is an *open* geometric morphism: the induced map of subobject posets $f^\star_A : \mathrm{Sub}_{\mathcal{F}}(A) \to \mathrm{Sub}_{\mathcal{E}}(f^\star(A))$ has a left adjoint. For sober topological spaces, this condition is equivalent to the assumption that the corresponding continuous function $f : X \to Y$ is an open map. [4, pp. 497, 537]

theory can be encoded as a set $\mathbb{T}$ of geometric sequents. The models of such a theory in the classical sense correspond exactly to the Σ-structures in **Set** that satisfy all of the geometric sequents of $\mathbb{T}$.[5]

Definition 2.14. A *geometric theory* over Σ is a set $\mathbb{T}$ of geometric sequents over Σ, called its *axioms*. A Σ-structure is said to be a *model* of $\mathbb{T}$ if it satisfies all of the axioms of $\mathbb{T}$. We write $\mathbb{T}\text{-Mod}(\mathcal{E})$ for the full subcategory of $\Sigma\text{-Str}(\mathcal{E})$ on the $\mathbb{T}$-models, so a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ induces a functor $\mathbb{T}\text{-Mod}(f^\star)$ making the following square commute.

$$\begin{tikzcd}
\mathbb{T}\text{-Mod}(\mathcal{F}) \arrow[r, dashed, "\mathbb{T}\text{-Mod}(f^\star)"] \arrow[d] & \mathbb{T}\text{-Mod}(\mathcal{E}) \arrow[d] \\
\Sigma\text{-Str}(\mathcal{F}) \arrow[r, "\Sigma\text{-Str}(f^\star)"'] & \Sigma\text{-Str}(\mathcal{E})
\end{tikzcd}$$

As in remark 2.4, we will occasionally write $\mathbb{T}\text{-Mod}(f^\star)$ as simply $f^\star$.

Examples 2.15.

(i) Let Σ_{Mon} be the structure with one sort A, and two function symbols

$$e : [] \to A \qquad\qquad m : [A, A] \to A$$

the latter of which we will represent with an infix dot. The geometric theory of monoids, $\mathbb{T}_{\text{Mon}}$, has axioms

$$\top \vdash_{[x,y,z]} (x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad \top \vdash_{[x]} x \cdot e = x \qquad \top \vdash_{[x]} e \cdot x = x$$

where the sort of each variable is A. The $\mathbb{T}_{\text{Mon}}$-models in **Set** are precisely the monoids.

(ii) Representing m with the symbol $+$, the theory of abelian groups $\mathbb{T}_{\text{AbGp}}$ can be obtained from $\mathbb{T}_{\text{Mon}}$ by adding the axioms

$$\top \vdash_{[x]} \exists y.\, x + y = e \qquad\qquad \top \vdash_{[x,y]} x + y = y + x$$

Let $\mathbb{Z}$ be the poset of integers, and let M be a $\mathbb{T}_{\text{AbGp}}$-model in $\mathbf{Psh}(\mathbb{Z})$. Then $P = MA$ is a presheaf on $\mathbb{Z}$, endowed with natural transformations

$$e : 1 \to P \qquad\qquad (+) : P \times P \to P$$

It is straightforward to verify that each of the Pn are abelian groups (in **Set**), with identity e_n and group operation $(+)_n$. Further, if the inequality $a \leq b$ in $\mathbb{Z}$ is witnessed by the morphism $h : a \to b$, the naturality conditions give the commutative squares

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "e_b"'] & 1 \arrow[d, "e_a"] \\
Pb \arrow[r, "Ph"'] & Pa
\end{tikzcd}
\qquad
\begin{tikzcd}
Pb \times Pb \arrow[r, "Ph \times Ph"] \arrow[d, "(+)_b"'] & Pa \times Pa \arrow[d, "(+)_a"] \\
Pb \arrow[r, "Ph"'] & Pa
\end{tikzcd}$$

Therefore, the maps $Ph : Pb \to Pa$ are group homomorphisms. Moreover, any chain complex of abelian groups $(A_n)_{n \in \mathbb{Z}}$ is naturally a $\mathbb{T}_{\text{AbGp}}$-model in $\mathbf{Psh}(\mathbb{Z})$; it can be viewed as an abelian group defined ‘over’ $\mathbb{Z}$.

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[5] Some sources, such as § X of [4], describe geometric logic as the logic of first-order formulas of the form $\forall \vec{x} : \vec{A}.\, \varphi(\vec{x}) \to \psi(\vec{x})$ where φ, ψ are geometric, instead of defining geometric sequents as in [2, p. 811].

(iii) Let $\mathbb{T}_{\mathrm{TorGp}}$ be the theory of torsion groups, which is obtained from $\mathbb{T}_{\mathrm{Mon}}$ by adding the axioms

$$\top \vdash_{[x]} \exists y.\, x \cdot y = e \qquad\qquad \top \vdash_{[x]} \bigvee_{n \geq 1} (x^n = e)$$

Note that a simple application of the compactness theorem shows that this cannot be written as a finitary first-order theory in the classical sense.

Let P be the presheaf on $\mathbb{N}^{\mathrm{op}}$ given by

$$Pa = \left\{ \exp\left(\frac{2\pi i t}{2^a}\right) \,\middle|\, t \in \mathbb{Z} \right\}$$

where the morphisms $Pa \to Pb$ for $a \leq b$ are the inclusion maps. P forms a $\mathbb{T}_{\mathrm{TorGp}}$-model in $\mathbf{Psh}(\mathbb{N}^{\mathrm{op}})$, where the group operation $P \times P \to P$ is pointwise multiplication. The only global element $1 \to P$ is the interpretation of the identity symbol $e : [] \to A$.

Now consider P as a presheaf over the discrete category $\mathrm{Discr}(\mathbb{N})$ with objects indexed by $\mathbb{N}$. Its global elements $1 \to P$ now correspond to elements of the abelian group $\prod_{a \in \mathbb{N}} Pa$, which is not torsion. Despite this, P also forms a $\mathbb{T}_{\mathrm{TorGp}}$-model over $\mathrm{Discr}(\mathbb{N})$, since it is 'locally' torsion. In general, there is an equivalence of categories

$$\mathbb{T}\text{-Mod}(\mathbf{Psh}(\mathcal{C})) \simeq [\mathcal{C}^{\mathrm{op}}, \mathbb{T}\text{-Mod}(\mathbf{Set})]$$

since terms and formulas are interpreted pointwise in presheaf categories. [2, p. 826]

(iv) Let $\mathbb{T}_{\mathrm{Ring}}$ be the theory of commutative unital rings, defined in the structure with one sort R and four function symbols

$$0 : [] \to R \qquad 1 : [] \to R \qquad (+) : [R, R] \to R \qquad (\cdot) : [R, R] \to R$$

Let X be a topological space, and let M be a $\mathbb{T}_{\mathrm{Ring}}$-model in $\mathbf{Sh}(X)$. Then $\mathcal{O}_X = MR$ is a sheaf of rings over X in the sense of algebraic geometry, and moreover, every sheaf of rings over X is naturally a $\mathbb{T}_{\mathrm{Ring}}$-model in $\mathbf{Sh}(X)$. [2, p. 827]

2.5 Classifying toposes and universal models

Theories, as syntax-based objects, cannot be easily manipulated using the semantics-based tools of category theory. To bypass this apparent difficulty, each geometric theory $\mathbb{T}$ will give rise to a particular category called its *classifying topos*, which encapsulates the syntactic properties of $\mathbb{T}$. To be more precise, models of $\mathbb{T}$ in an arbitrary Grothendieck topos $\mathcal{E}$ will correspond to geometric morphisms from $\mathcal{E}$ to the classifying topos. Therefore, the classifying topos for $\mathbb{T}$ is a representing object for the functor that maps a Grothendieck topos $\mathcal{E}$ to its category of models of $\mathbb{T}$.

Definition 2.16. Let $\mathbb{T}$ be a geometric theory. A Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ is called a *classifying topos* for $\mathbb{T}$ if, for every Grothendieck topos $\mathcal{E}$, there is an equivalence of categories

$$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathfrak{Top}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$$

If $\mathbf{Set}[\mathbb{T}]$ is a classifying topos for $\mathbb{T}$, then the $\mathbb{T}$-model in $\mathbf{Set}[\mathbb{T}]$ that corresponds to the geometric morphism $1 : \mathbf{Set}[\mathbb{T}] \to \mathbf{Set}[\mathbb{T}]$ is called the *universal model* $U_{\mathbb{T}}$. The universal model is a 'generic' model of $\mathbb{T}$ in a number of useful ways, for example, it satisfies precisely those geometric sequents that hold in all Grothendieck toposes. We will use this fact to prove the completeness theorem for geometric logic.

In a precise sense, the Grothendieck topos **Set**$[\mathbb{T}]$ looks like a copy of **Set** that is *forced* to contain a universal model of $\mathbb{T}$: the objects and morphisms in **Set**$[\mathbb{T}]$ are precisely those that must exist in any Grothendieck topos containing a model of $\mathbb{T}$. The link with forcing in model theory is explored in more detail in [5].

Since they are defined by a universal property, classifying toposes are unique up to equivalence.

Examples 2.17.

(i) We claim that **Set** is the classifying topos for the empty theory $\mathbb{T} = \varnothing$ over the empty signature. There is a unique $\mathbb{T}$-model in any Grothendieck topos $\mathcal{E}$, so $\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathbf{1}$, the terminal category. Every Grothendieck topos $\mathcal{E}$ admits a geometric morphism to **Set** whose inverse image $f^\star$ is given by mapping a set x to the x-indexed copower of 1 in $\mathcal{E}$. This geometric morphism is unique up to isomorphism, so we also have $\mathfrak{Top}(\mathcal{E}, \mathbf{Set}) \simeq \mathbf{1}$. We briefly remark that since every Grothendieck topos admits a geometric morphism to **Set**, any geometric theory that is classically consistent (that is, has a model in **Set**) has a model in every Grothendieck topos by taking its inverse image.

(ii) Let $\mathbb{T}$ be the inconsistent theory $\{\top \vdash_{[]} \bot\}$ over the empty signature. We claim that $\mathbf{1} = \mathbf{Psh}(\mathbf{0})$ is the classifying topos for $\mathbb{T}$, where $\mathbf{0}$ is the initial category. Every nontrivial Grothendieck topos has no $\mathbb{T}$-models and admits no geometric morphisms to $\mathbf{Psh}(\mathbf{0})$, and $\mathbf{Psh}(\mathbf{0})$ itself contains precisely one $\mathbb{T}$-model (the universal model) and has only the identity geometric morphism to itself. Some authors refer to $\mathbf{Psh}(\mathbf{0})$ as the *inconsistent topos*.

(iii) Let Σ be the signature with one sort A and no function or relation symbols, and let $\mathbb{T}$ be the theory over Σ with no axioms. This is called the *geometric theory of objects*, since models in a Grothendieck topos $\mathcal{E}$ correspond to the objects of $\mathcal{E}$. One can show that $\mathbf{Psh}(\mathbf{Fin}^{\mathrm{op}})$ is the classifying topos for $\mathbb{T}$, where **Fin** is the category of finite sets.[6] The presheaf $\mathbf{Y}1$ is the universal model of $\mathbb{T}$, where $\mathbf{Y}$ is the Yoneda embedding from remark 1.12 (ii).

2.6 The classifying topos for rings

We will now present an extended worked example from [4, p. 439]. Let $\mathbb{T}_{\mathrm{Ring}}$ be the theory of rings, as in example 2.15 (iv). We say that a ring (in **Set**) is *finitely presented* if it is isomorphic to

$$\frac{\mathbb{Z}[X_1, \dots, X_n]}{(P_1, \dots, P_k)}$$

where the P_i are polynomials in the indeterminates $X_1, \dots, X_n$. We will show that $\mathbf{Psh}(\mathbf{FpRing}^{\mathrm{op}})$ is the classifying topos of $\mathbb{T}_{\mathrm{Ring}}$ where **FpRing** is the category of finitely presented rings,[7] and that the universal model is $\mathbf{Y}(\mathbb{Z}[X])$.

To show this, we will use without proof the fact that there is an equivalence of categories

$$\mathfrak{Top}(\mathcal{E}, \mathbf{Psh}(\mathbf{FpRing}^{\mathrm{op}})) \simeq \mathbf{Flat}(\mathbf{FpRing}^{\mathrm{op}}, \mathcal{E})$$

where $\mathbf{Flat}(\mathbf{FpRing}^{\mathrm{op}}, \mathcal{E})$ is the category of functors $\mathbf{FpRing}^{\mathrm{op}} \to \mathcal{E}$ that preserve finite limits.[8] The forward direction of this equivalence is given on objects by mapping a geometric morphism f to the functor $f^\star\mathbf{Y}$. Note that **FpRing** has all finite colimits, where the coproducts are tensor products, and the coequalisers are quotient rings. We will explore this correspondence in more generality in § 4.2.

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[6] More precisely, our definitions require **Fin** to be a small category equivalent to the category of finite sets, such as V_ω.

[7] Again, we should technically choose a small category equivalent to **FpRing**.

[8] There is a more general definition of *flat* functor $\mathcal{C} \to \mathcal{E}$, discussed in [4, p. 392]. By corollary VII.10.3 of [4, p. 409], if $\mathcal{C}$ has finite limits and $\mathcal{E}$ is a Grothendieck topos, then a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.

We will now show that functors $\mathbf{FpRing}^{\mathrm{op}} \to \mathcal{E}$ that preserve finite limits correspond to ring objects in $\mathcal{E}$. To avoid confusion, we will only write morphisms in terms of **FpRing** and not $\mathbf{FpRing}^{\mathrm{op}}$.

First, given such a functor F, there is an object $F(\mathbb{Z}[X])$ of $\mathcal{E}$. All of the algebraic structure of $\mathbb{Z}[X]$ can be pushed along F since it preserves finite limits, thus turning $F(\mathbb{Z}[X])$ into a $\mathbb{T}_{\mathrm{Ring}}$-model.

Conversely, if we have a $\mathbb{T}_{\mathrm{Ring}}$-model R in $\mathcal{E}$, there is an *evaluation functor* $\mathrm{ev}_R : \mathbf{FpRing}^{\mathrm{op}} \to \mathcal{E}$ given as follows. First, we must set $\mathrm{ev}_R(\mathbb{Z}[X]) = R$ if the two constructions are to be inverses. In order to force ev_R to preserve finite limits, it must map tensor products in **FpRing** to products in $\mathcal{E}$. Therefore,

$$\mathrm{ev}_R(\mathbb{Z}[X_1, \dots, X_n]) = R^n$$

Before completing the definition of ev_R on objects, we briefly discuss its action on morphisms of the form

$$f : \mathbb{Z}[X_1, \dots, X_n] \to \mathbb{Z}[Y_1, \dots, Y_m]$$

Such a morphism is given by a sequence of polynomials $P_1, \dots, P_n$ in the Y_i with coefficients in $\mathbb{Z}$. Any such polynomial P corresponds to a term t_P of sort $R^m \to R$ in the language of rings (assuming we have a negation operation), and so it has an interpretation as a morphism $\llbracket [\,] \,.\, t_P \rrbracket : R^m \to R$ in $\mathcal{E}$. We can therefore define $\mathrm{ev}_R(f) : R^m \to R^n$ by

$$\pi_i \, \mathrm{ev}_R(f) = \llbracket [\,] \,.\, t_{P_i} \rrbracket$$

Now, ev_R must map quotient rings to equalisers in $\mathcal{E}$ given by the interpretation of the polynomials. More precisely, we must have an equaliser diagram

$$\begin{tikzcd} \mathrm{ev}_R\left(\frac{\mathbb{Z}[X_1,\dots,X_n]}{(P_1,\dots,P_k)}\right) \arrow[r] & R^n \arrow[r, shift left, "(\llbracket P_1 \rrbracket,\dots,\llbracket P_k \rrbracket)"] \arrow[r, shift right, "(\llbracket 0 \rrbracket,\dots,\llbracket 0 \rrbracket)"'] & R^k \end{tikzcd}$$

We can use this to define ev_R on all objects of $\mathbf{FpRing}^{\mathrm{op}}$. Finally, this construction allows ev_R to be extended in a natural way to morphisms between finitely presented rings.

The two constructions can be seen to be inverses. If R is a $\mathbb{T}_{\mathrm{Ring}}$-model in $\mathcal{E}$, then by construction, there is an isomorphism of $\mathbb{T}_{\mathrm{Ring}}$-models $R \cong \mathrm{ev}_R \, \mathbb{Z}[X]$, natural in R. In the other direction, if $F : \mathbf{FpRing}^{\mathrm{op}} \to \mathcal{E}$ is a functor that preserves finite limits, then there is a natural isomorphism $\alpha_F : F \to \mathrm{ev}_{F(\mathbb{Z}[X])}$. Its components on polynomial rings are morphisms

$$(\alpha_F)_{\mathbb{Z}[X_1,\dots,X_n]} : F(\mathbb{Z}[X_1, \dots, X_n]) \to \mathrm{ev}_{F(\mathbb{Z}[X])}(\mathbb{Z}[X_1, \dots, X_n])$$

We have isomorphisms

$$F(\mathbb{Z}[X_1, \dots, X_n]) \cong F(\mathbb{Z}[X])^n \cong \mathrm{ev}_{F(\mathbb{Z}[X])}(\mathbb{Z}[X_1, \dots, X_n])$$

which are natural in the parameter, and so they can be used to define α_F. A similar calculation allows us to extend α_F to all finitely presented rings. But this construction is natural in F, so we obtain the desired equivalence of categories.

More examples can be found in [4, pp. 419–466].

3 Geometric logic and geometric categories

We will now briefly describe the relevant notion of syntactic proof, called geometric logic.

3.1 Geometric logic

Our logic is presented as a calculus for geometric sequents over a fixed signature Σ, with rules written in the form

$$\frac{(\varphi_i \vdash_{\vec{x}_i} \psi_i)_{i\in I}}{\varphi \vdash_{\vec{x}} \psi}$$

Such a rule states that the sequent below the line may be proven by supplying proofs for the sequents above the line. If there are no sequents above the line, the sequent below may be called an axiom of geometric logic. The standard structural rules of sequent calculus (identity, substitution, cut, and weakening) are assumed; for details, see [2, p. 830]. The rules for existential quantification are

$$\frac{\varphi \vdash_{[\vec{x},y:A]} \psi}{\exists y : A.\,\varphi \vdash_{\vec{x}} \psi} \qquad \frac{\exists y : A.\,\varphi \vdash_{\vec{x}} \psi}{\varphi \vdash_{[\vec{x},y:A]} \psi} \qquad \frac{}{\varphi \wedge (\exists y : A.\,\chi) \vdash_{\vec{x}} \exists y : A.\,(\varphi \wedge \chi)}$$

where y does not appear free in ψ and does not occur in the context $\vec{x}$. The third rule, called the *Frobenius axiom*, is provable in first-order logic with implication, but not from only the other rules of geometric logic. The rules for disjunction are

$$\frac{}{\varphi_i \vdash_{\vec{x}} \bigvee_{i\in I} \varphi_i} \qquad \frac{(\varphi_i \vdash_{\vec{x}} \psi)_{i\in I}}{\bigvee_{i\in I} \varphi_i \vdash_{\vec{x}} \psi} \qquad \frac{}{\varphi \wedge \bigvee_{i\in I} \psi_i \vdash_{\vec{x}} \bigvee_{i\in I} (\varphi \wedge \psi_i)}$$

The last axiom, called the *distributive axiom*, is not provable from the other rules for similar reasons to the Frobenius axiom. However, the converses of both of these axioms are derivable in geometric logic.

The remaining rules for the basic logical symbols are the same as in finitary first-order intuitionistic logic, so for example we do not assume the law of the excluded middle. Note that because of the presence of infinitary disjunctions, proof trees may be infinite, but are required to be well-founded. For a detailed reference, see either [2, p. 830] or the latter half of the Part III lecture course *Model Theory and Non-Classical Logic* [6].

Theorem 3.1 (soundness). Let $\mathbb{T}$ be a geometric theory. If a sequent $\varphi \vdash_{\vec{x}} \psi$ is derivable in geometric logic, taking the sequents in $\mathbb{T}$ as additional axioms, then $\varphi \vdash_{\vec{x}} \psi$ is satisfied in every model of $\mathbb{T}$ in every Grothendieck topos.

Proof sketch. It suffices by induction to verify that the deduction rules preserve satisfaction. In most cases, this verification is straightforward. The Frobenius and distributive axioms follow from the fact that images and unions are stable under pullback in Grothendieck toposes. [2, p. 832] □

3.2 The syntactic category

For a fixed geometric theory $\mathbb{T}$, we will build a category $\mathcal{C}_{\mathbb{T}}$ out of the syntax of $\mathbb{T}$, and use this to prove a completeness theorem for geometric logic. We will do this differently to the usual construction of term models from model theory. When building term models, we must repeatedly extend a theory to a maximal consistent set of sentences and add constants witnessing existentials. This process is required by the fact that truth values in models in the classical sense must take one of only two fixed values. Conversely, when constructing the syntactic category of a geometric theory, we need only define the entailment relation between sentences (which will be provability in geometric logic), and need not fix a choice of truth value for a given sentence.

The main idea is that the objects and morphisms of $\mathcal{C}_{\mathbb{T}}$ will correspond to those objects and morphisms that we can prove to exist from $\mathbb{T}$. More precisely, given a model M of $\mathbb{T}$ in some Grothendieck topos, any object of $\mathcal{C}_{\mathbb{T}}$ will correspond to the subobject of some $M\vec{A}$ defined by some formula-in-context $\vec{x} : \vec{A} \,.\, \varphi$. Note that the only objects of a Grothendieck topos that are definable using only geometric logic in a structure M are of this form; for instance, we cannot talk directly about power objects or colimits in geometric logic, so they will not appear in syntactic categories. The morphisms between objects in $\mathcal{C}_{\mathbb{T}}$ will correspond to morphisms between the relevant subobjects in any model M. In this way, every model M in a Grothendieck topos $\mathcal{E}$ has a copy of $\mathcal{C}_{\mathbb{T}}$ inside it, and so each such model will give rise to an *evaluation functor* $\mathrm{ev}_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ which realises the syntax inside $\mathcal{C}_{\mathbb{T}}$. These behave similarly to the evaluation functors for rings defined in § 2.6. We will later show in theorem 3.14 that all suitably nice functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ are evaluation functors for some model.

This category $\mathcal{C}_{\mathbb{T}}$ will not turn out to be a Grothendieck topos (in fact, it will be small), but it will satisfy enough of the properties of Grothendieck toposes that we can still interpret terms and formulas in it. We will show that it contains a generic model $M_{\mathbb{T}}$ of $\mathbb{T}$ that satisfies precisely those sequents that $\mathbb{T}$ proves. Later, we will endow $\mathcal{C}_{\mathbb{T}}$ with a suitable Grothendieck coverage J to construct a Grothendieck topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$. This will turn out to be the classifying topos of $\mathbb{T}$, and the image of the generic model $M_{\mathbb{T}}$ under the Yoneda embedding $\mathbf{AY} : \mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$ will turn out to be the universal model $U_{\mathbb{T}}$.

Definition 3.2. Let $\mathbb{T}$ be a geometric theory. We define its *syntactic category* $\mathcal{C}_{\mathbb{T}}$ as follows.

(i) We say that formulas-in-context $\vec{x} \,.\, \varphi$ and $\vec{y} \,.\, \psi$ are *α-equivalent* if they are equal up to renaming the bound variables and the variables in the context. Note that the interpretations of α-equivalent formulas-in-context in any model of $\mathbb{T}$ are equal. The objects of $\mathcal{C}_{\mathbb{T}}$ are defined to be the α-equivalence classes of geometric formulas-in-context, where we denote the equivalence class of $\vec{x} \,.\, \varphi$ by $\{\vec{x} \,.\, \varphi\}$.

(ii) Let $\{\vec{x} \,.\, \varphi\}, \{\vec{y} \,.\, \psi\}$ be objects of $\mathcal{C}_{\mathbb{T}}$, where without loss of generality we take $\vec{x}$ and $\vec{y}$ to be disjoint. A $\mathbb{T}$-*graph* from $\{\vec{x} \,.\, \varphi\}$ to $\{\vec{y} \,.\, \psi\}$ is a geometric formula-in-context $[\vec{x}, \vec{y}] \,.\, \theta$ such that $\mathbb{T}$ proves the sequents

$$\theta(\vec{x}, \vec{y}) \vdash_{[\vec{x},\vec{y}]} \varphi(\vec{x}) \wedge \psi(\vec{y}) \qquad \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{z}) \vdash_{[\vec{x},\vec{y},\vec{z}]} \vec{y} = \vec{z} \qquad \varphi(\vec{x}) \vdash_{[\vec{x}]} \exists \vec{y}.\, \theta(\vec{x}, \vec{y})$$

The morphisms $\{\vec{x} \,.\, \varphi\} \to \{\vec{y} \,.\, \psi\}$ are the equivalence classes of $\mathbb{T}$-graphs, where two graphs θ, η are regarded as equal if $\mathbb{T}$ proves they are the same; that is, $\mathbb{T}$ proves the sequents

$$\theta(\vec{x}, \vec{y}) \vdash_{[\vec{x},\vec{y}]} \eta(\vec{x}, \vec{y}) \qquad \eta(\vec{x}, \vec{y}) \vdash_{[\vec{x},\vec{y}]} \theta(\vec{x}, \vec{y})$$

The equivalence class of a graph $[\vec{x}, \vec{y}] \,.\, \theta$ is denoted $[\theta]$.

Remark 3.3. Before developing the syntactic category, we will first demonstrate that the notion of $\mathbb{T}$-graph is sensible. Let $f : A \to B$ be a morphism in some Grothendieck topos. The *graph* of f is the subobject G of $A \times B$ given by the equaliser of

$$\begin{tikzcd} A \times B \arrow[r, shift left, "f\pi_1"] \arrow[r, shift right, "\pi_2"'] & B \end{tikzcd}$$

Given such a G, we can reconstruct the original morphism. The image of the composite

$$G \xrightarrow{g} [\![\vec{x} \,.\, \varphi]\!] \times [\![\vec{y} \,.\, \psi]\!] \xrightarrow{\pi_1} [\![\vec{x} \,.\, \varphi]\!]$$

is the maximal subobject, so the morphism $\pi_1 g : G \to A$ is epic; one can show in general that a morphism in a Grothendieck topos is epic if and only if its image is maximal. But it is clearly also monic, so it has an inverse h, and so we can reconstruct f as $f = \pi_2 gh$. This can be viewed as a form of *definite description* internal to $\mathcal{E}$: every graph is the graph of some function. The facts about G that we used to reconstruct f correspond exactly to the second and third axioms of $\mathbb{T}$-graphs in definition 3.2 above.

Proposition 3.4. $\mathcal{C}_{\mathbb{T}}$ is a category.

Proof. The identity

$$\{\vec{x}\,.\,\varphi\} \xrightarrow{[\theta]} \{\vec{y}\,.\,\varphi\}$$

for disjoint $\vec{x}, \vec{y}$ is given by

$$\theta(\vec{x}, \vec{y}) :\equiv (\vec{x} = \vec{y} \wedge \varphi(\vec{x}))$$

The composition of morphisms

$$\{\vec{x}\,.\,\varphi\} \xrightarrow{[\theta]} \{\vec{y}\,.\,\psi\} \xrightarrow{[\eta]} \{\vec{z}\,.\,\chi\}$$

is given by

$$(\eta \circ \theta)(\vec{x}, \vec{z}) :\equiv (\exists \vec{y}.\, \theta(\vec{x}, \vec{y}) \wedge \eta(\vec{y}, \vec{z}))$$

Composition is not strictly associative, but $\mathbb{T}$ proves that the two composites are equal. □

Remark 3.5. If Σ has no sorts, a theory $\mathbb{T}$ over Σ is called a *propositional* theory. In this case, $\mathcal{C}_{\mathbb{T}}$ is the Lindenbaum–Tarski algebra of $\mathbb{T}$ with respect to geometric logic. [2, p. 851]

3.3 Geometric categories

When interpreting terms and formulas in a Grothendieck topos $\mathcal{E}$, we only used the facts that

- $\mathcal{E}$ has finite limits;
- $\mathcal{E}$ has coequalisers of kernel pairs (image factorisations) and these factorisations are stable under pullback;
- subobject posets in $\mathcal{E}$ have small unions and these unions are stable under pullback.

When studying the induced functor between categories of $\mathbb{T}$-models arising from a geometric morphism $f = (f^\star \dashv f_\star)$, we only used the facts that

- $f^\star$ preserves finite limits;
- $f^\star$ preserves regular epimorphisms (and hence image factorisations);
- $f^\star$ preserves small unions of subobjects.

Motivated by this, we define the more general notions of *geometric category* and *geometric functor*.

Definition 3.6. [2, pp. 43, 821] A well-powered category is called *geometric* if

(i) it has all finite limits;

(ii) the kernel pair of any morphism has a coequaliser, and coequalisers of kernel pairs are stable under pullback;

(iii) the subobject posets have all small unions, and these unions are stable under pullback.

A functor between geometric categories is called *geometric* if

(i) it preserves all finite limits;

(ii) it preserves regular epimorphisms (and hence image factorisations);

(iii) it preserves small unions of subobjects.

We write $\mathfrak{Geom}$ for the 2-category of geometric categories, geometric functors, and the natural transformations between them.

Therefore, we can define interpretations of terms and formulas in any geometric category, and geometric functors between such categories preserve satisfaction of geometric sequents. Our proof of soundness in theorem 3.1 works identically in geometric categories, so statements provable in geometric logic hold in every geometric category. Similarly, by following the proofs, it is easy to verify that all geometric categories are balanced, and satisfy the form of definite description in remark 3.3.

Lemma 3.7. $\mathcal{C}_{\mathbb{T}}$ is a geometric category.

Proof. Part (i): finite limits. The terminal object is $\{[\,]\,.\,\top\}$. Let $\{\vec{x}\,.\,\varphi\}$ and $\{\vec{y}\,.\,\psi\}$ be objects in $\mathcal{C}_{\mathbb{T}}$, where without loss of generality we take $\vec{x}, \vec{y}$ to be distinct. Then their product is

$$\{[\vec{x}, \vec{y}]\,.\,\varphi \wedge \psi\}$$

If $[\theta], [\eta] : \{\vec{x}\,.\,\varphi\} \rightrightarrows \{\vec{y}\,.\,\psi\}$, their equaliser is the obvious map

$$\{\vec{x}\,.\,\exists\vec{y}.\,(\theta \wedge \eta)\} \longrightarrow \{\vec{x}\,.\,\varphi\}$$

These three constructions suffice to generate all finite limits by [3, p. 113], but it is useful to also state the result directly for pullbacks, which take the form

$$\begin{tikzcd}
\{[\vec{x}, \vec{y}]\,.\,\exists\vec{z}.\,(\theta \wedge \eta)\} \arrow[r] \arrow[d] & \{\vec{x}\,.\,\varphi\} \arrow[d, "{[\theta]}"] \\
\{\vec{y}\,.\,\psi\} \arrow[r, "{[\eta]}"'] & \{\vec{z}\,.\,\chi\}
\end{tikzcd}$$

where $\vec{x}, \vec{y}, \vec{z}$ are disjoint.

Part (ii): image factorisations. The image factorisation of $[\theta] : \{\vec{x}\,.\,\varphi\} \to \{\vec{y}\,.\,\psi\}$ is

$$\begin{tikzcd}
\{\vec{x}\,.\,\varphi\} \arrow[r, "{[\theta]}"] \arrow[dr, "{[\theta]}"'] & \{\vec{y}\,.\,\exists\vec{x}.\,\theta\} \arrow[d] \\
 & \{\vec{y}\,.\,\psi\}
\end{tikzcd}$$

The stability of this factorisation under pullback can be easily checked using the Frobenius axiom.

Part (iii): small unions. Let $\left(\{(\vec{y})_i\,.\,\psi_i\}\right)_{i\in I}$ be subobjects of $\{\vec{x}\,.\,\varphi\}$. Without loss of generality, we can set each $(\vec{y})_i$ to be $\vec{x}$ by taking images. Then, we simply have

$$\bigcup_{i\in I}\{\vec{x}\,.\,\psi_i\} = \left\{\vec{x}\,.\,\bigvee_{i\in I}\psi_i\right\}$$

Stability under pullback follows from the distributive axiom. ☐

It will be useful to note the following general fact about geometric categories.

Proposition 3.8. Every epimorphism in a geometric category $\mathcal{C}$ is regular.

One part of the definition of a geometric functor F was that it must preserve regular epimorphisms; with this proposition, we may instead prove that F preserves epimorphisms.

Proof. [2, p. 22] Let $f : A \to B$ be an epimorphism in $\mathcal{C}$, and let $(a, b) : R \rightrightarrows A$ be its kernel pair. We claim that f is the coequaliser of (a, b). Let $c : A \to C$ be a morphism with $ca = cb$, and consider the image factorisation

$$\begin{tikzcd}
A \arrow[r, two heads, "d"] \arrow[dr, "{(f,c)}"'] & D \arrow[d, tail, "{(g,h)}"] \\
& B \times C
\end{tikzcd}$$

We will show that g is an isomorphism. Then, $hg^{-1} : B \to C$ gives rise to the commutative triangle

$$\begin{tikzcd}
A \arrow[d, two heads, "f"'] \arrow[dr, "c"] & \\
B \arrow[r, "hg^{-1}"'] & C
\end{tikzcd}$$

thus exhibiting a factorisation of c through f.

First, note that g is epic as $gd = f$. We show that g is monic; this suffices as geometric categories are balanced. Let $k, l : E \rightrightarrows D$ be such that $gk = gl$; we first aim to show that $hk = hl$. Form the pullback of $(k, l) : E \to D \times D$ along $d \times d : A \times A \to D \times D$.

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "{(m,n)}"'] & E \arrow[d, "{(k,l)}"] \\
A \times A \arrow[r, "d \times d"'] & D \times D
\end{tikzcd}$$

Note that the morphism $d \times d : A \times A \to D \times D$ is epic and so has maximal image, but as images are stable under pullback, p also has maximal image, and so is epic. Since $gk = gl$, we have

$$fm = gdm = gkp = glp = gdn = fn$$

Since (a, b) is the kernel pair of f, we have a morphism q making the following diagram commute.

$$\begin{tikzcd}
P \arrow[dr, dashed, "q"] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "n"'] & & \\
& R \arrow[r, "a"] \arrow[d, "b"'] & A \arrow[d, two heads, "f"] \\
& A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

Therefore, as $ca = cb$,

$$hkp = hdm = cm = caq = cbq = cn = hdn = hlp$$

Therefore $hk = hl$ as claimed. But then $(g, h)k = (g, h)l$, so $k = l$ as (g, h) is monic. Therefore, g is monic, and hence an isomorphism, giving the result. ☐

3.4 The generic model

We now build the ‘model from syntax’ of $\mathbb{T}$ as claimed.

Definition 3.9. The *generic model* of a geometric theory $\mathbb{T}$ is the Σ-structure $M_{\mathbb{T}}$ in $\mathcal{C}_{\mathbb{T}}$ given by

$$M_{\mathbb{T}}A \coloneqq \{[x : A] \,.\, \top\}$$

where

(i) if $f : \vec{A} \to B$ is a function symbol in Σ, then $M_{\mathbb{T}}f$ is

$$\left\{\vec{x} : \vec{A} \,.\, \top\right\} \xrightarrow{[f(\vec{x})=y]} \{[y : B] \,.\, \top\}$$

where $\vec{x}, y$ are disjoint.

(ii) if $R \rightarrowtail \vec{A}$ is a relation symbol in Σ, then $M_{\mathbb{T}}R$ is

$$\left\{\vec{x} : \vec{A} \,.\, R(\vec{x})\right\} \xrightarrow{[\vec{x}=\vec{y} \wedge R(\vec{x})]} \left\{\vec{y} : \vec{A} \,.\, \top\right\}$$

where $\vec{x}, \vec{y}$ are disjoint.

Lemma 3.10. For every geometric sequent $\varphi \vdash_{\vec{x}} \psi$,

$$(\mathbb{T} \text{ proves } \varphi \vdash_{\vec{x}} \psi) \leftrightarrow (M_{\mathbb{T}} \text{ satisfies } \varphi \vdash_{\vec{x}} \psi)$$

In particular, $M_{\mathbb{T}}$ is a model of $\mathbb{T}$.

Proof. Suppose that $\mathbb{T}$ proves $\varphi \vdash_{\vec{x}} \psi$. Then we have a morphism

$$\left\{\vec{x} \,.\, \varphi(\vec{x})\right\} \xrightarrow{[\vec{x}=\vec{y} \wedge \varphi(\vec{x})]} \left\{\vec{y} \,.\, \psi(\vec{y})\right\}$$

as required. Conversely, if $\left\{\vec{x} \,.\, \varphi\right\} \leq \left\{\vec{x} \,.\, \psi\right\}$ as subobjects of $\left\{\vec{x} \,.\, \top\right\}$, the morphism $\left\{\vec{x} \,.\, \varphi\right\} \to \left\{\vec{x} \,.\, \psi\right\}$ witnesses a $\mathbb{T}$-proof of the sequent $\varphi \vdash_{\vec{x}} \psi$ as required. ☐

This establishes one form of the completeness theorem for geometric logic.

Corollary 3.11 (completeness in geometric categories). For every geometric sequent $\varphi \vdash_{\vec{x}} \psi$,

$$(\mathbb{T} \text{ proves } \varphi \vdash_{\vec{x}} \psi) \leftrightarrow (\text{every model of } \mathbb{T} \text{ in a geometric category satisfies } \varphi \vdash_{\vec{x}} \psi)$$

Later, in corollary 4.5, we will show a strengthening of this result:

$$(\mathbb{T} \text{ proves } \varphi \vdash_{\vec{x}} \psi) \leftrightarrow (\text{every model of } \mathbb{T} \text{ in a Grothendieck topos satisfies } \varphi \vdash_{\vec{x}} \psi)$$

As indicated above, we will do this by turning $\mathcal{C}_{\mathbb{T}}$ into a Grothendieck topos.

We will now prove one of our obligations from § 3.2, namely, that every model M of $\mathbb{T}$ in a Grothendieck topos has a copy of $\mathcal{C}_{\mathbb{T}}$ inside it.

Definition 3.12. Let M be a model of $\mathbb{T}$ in a geometric category $\mathcal{E}$. The *evaluation functor* $\mathrm{ev}_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ is given on objects by

$$\mathrm{ev}_M \{\vec{x} \,.\, \varphi\} = [\![\vec{x} \,.\, \varphi]\!]_M$$

and if $\theta : \{\vec{x} \,.\, \varphi\} \to \{\vec{y} \,.\, \psi\}$, then the interpretation of θ in M is the graph of a unique function $f : [\![\vec{x} \,.\, \varphi]\!]_M \to [\![\vec{y} \,.\, \psi]\!]_M$, and so we define $\mathrm{ev}_M[\theta] = f$. Functoriality of ev_M follows from the definition of composition in $\mathcal{C}_{\mathbb{T}}$.

We will now show that the evaluation functor behaves like the evaluation functors from § 2.6. We showed that evaluation functors for rings preserve finite limits, and that there is an equivalence of categories between $\mathbb{T}_{\mathrm{Ring}}$-Mod($\mathcal{E}$) and the full subcategory of functors $\mathbf{FpRing}^{\mathrm{op}} \to \mathcal{E}$ that preserve finite limits. We can prove the same results in similar ways, replacing 'preserves finite limits' with 'geometric' wherever necessary.

Theorem 3.13. For any $\mathbb{T}$-model M in a geometric category $\mathcal{E}$, the evaluation functor ev_M is geometric.

As geometric functors preserve models, this theorem implies that every model M is of the form $\mathrm{ev}_M\, M_{\mathbb{T}}$, where $M_{\mathbb{T}}$ is the generic model.

Proof. First, we show ev_M preserves image factorisations. Recall that the image factorisation of $[\theta] : \{\vec{x} \,.\, \varphi\} \to \{\vec{y} \,.\, \psi\}$ in $\mathcal{C}_{\mathbb{T}}$ is

$$\begin{tikzcd}
\{\vec{x} \,.\, \varphi\} \arrow[r, "{[\theta]}"] \arrow[dr, "{[\theta]}"'] & \{\vec{y} \,.\, \exists \vec{x}.\, \theta\} \arrow[d] \\
 & \{\vec{y} \,.\, \psi\}
\end{tikzcd}$$

But this is mapped to an image factorisation under ev_M by the definition of the interpretation of the existential. It is clear that small unions are preserved by ev_M.

It now suffices to show that it preserves finite limits. For finite products,

$$\mathrm{ev}_M \{\vec{x}, \vec{y} \,.\, \varphi \wedge \psi\} = [\![\vec{x}, \vec{y} \,.\, \varphi \wedge \psi]\!] \cong [\![\vec{x} \,.\, \varphi]\!] \times [\![\vec{y} \,.\, \psi]\!]$$

and the result for the terminal object is trivial. For equalisers, let $[\theta], [\eta] : \{\vec{x} \,.\, \varphi\} \rightrightarrows \{\vec{y} \,.\, \psi\}$ be such that θ, η are the graphs of morphisms f, g in $\mathcal{E}$. Let m be the subobject inclusion

$$[\![\vec{x} \,.\, \exists \vec{y}.\, (\theta \wedge \eta)]\!] \rightarrowtail [\![\vec{x} \,.\, \varphi]\!]$$

Then we have $fm = gm$. This is because the morphisms in the following square are $\mathbb{T}$-provably equivalent, and ev_M is functorial.

$$\begin{tikzcd}
\{\vec{x} \,.\, \exists \vec{y}.\, (\theta \wedge \eta)\} \arrow[r, tail] \arrow[d, tail] & \{\vec{x} \,.\, \varphi\} \arrow[d, "{[\theta]}"] \\
\{\vec{x} \,.\, \varphi\} \arrow[r, "{[\eta]}"'] & \{\vec{y} \,.\, \psi\}
\end{tikzcd}$$

But if $a : A \to [\![\vec{x} \,.\, \varphi]\!]$ has $fa = ga$, then we have a factorisation

$$\begin{tikzcd}
 & {[\![[\vec{x}, \vec{y}] \,.\, \theta \wedge \eta]\!]} \arrow[d] \\
A \arrow[ur, dashed] \arrow[r, "{(a, fa)}"'] & {[\![\vec{x} \,.\, \varphi]\!] \times [\![\vec{y} \,.\, \psi]\!]}
\end{tikzcd}$$

which gives rise to a factorisation of a through m by the definition of the interpretation of the existential. This shows that ev_M is a geometric functor, as required. $\square$

Let $\mathcal{E}$ be a geometric category and let $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ be a geometric functor. Then, $FM_{\mathbb{T}}$ is a $\mathbb{T}$-model in $\mathcal{E}$, because geometric functors preserve models of geometric theories. Because $M_{\mathbb{T}}$ is made purely out of syntax, every model in a geometric category arises in this way, where F is the evaluation functor ev_M.

Theorem 3.14. For each geometric category $\mathcal{E}$, there is an equivalence of categories

$$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathfrak{Geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$$

Proof. In the forward direction, a model M is mapped to its evaluation functor ev_M. This is functorial, because for every morphism of models $M \to N$, we obtain morphisms $[\![\vec{x} \,.\, \varphi]\!]_M \to [\![\vec{x} \,.\, \varphi]\!]_N$ in $\mathcal{E}$, which give rise to a natural transformation $\mathrm{ev}_M \to \mathrm{ev}_N$. In the backward direction, a geometric functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ is mapped to the model $FM_{\mathbb{T}}$. This is also functorial, as a natural transformation of such functors $\alpha : F \to G$ gives rise to a morphism of Σ-structures $h : FM_{\mathbb{T}} \to GM_{\mathbb{T}}$, given on each sort A by $h_A = \alpha_{M_{\mathbb{T}}A}$.

If M is a $\mathbb{T}$-model in $\mathcal{E}$, we have an isomorphism $M \cong \mathrm{ev}_M M_{\mathbb{T}}$, which is clearly natural in M. It remains to show that if $F : \mathcal{C}_{\mathbb{T}} \to E$ is geometric, then F is naturally isomorphic to $\mathrm{ev}_{FM_{\mathbb{T}}}$, and that this isomorphism is natural in F. To define a natural transformation $\alpha_F : F \to \mathrm{ev}_{FM_{\mathbb{T}}}$, we must define components

$$(\alpha_F)_{\{\vec{x} \,.\, \varphi\}} : F\{\vec{x} \,.\, \varphi\} \to [\![\vec{x} \,.\, \varphi]\!]_{FM_{\mathbb{T}}}$$

As F is geometric, we can easily produce isomorphisms $F\{\vec{x} \,.\, \varphi\} \cong [\![\vec{x} \,.\, \varphi]\!]_{FM_{\mathbb{T}}}$ by induction on φ, and we can use these as the components for α_F. The construction of these isomorphisms is natural in F, due to the naturality of interpretations of terms and formulas. $\square$

4 Toposes from syntax

4.1 The canonical coverage

In this section, we will discuss how we can turn the syntactic category $\mathcal{C}_{\mathbb{T}}$ into the site for a Grothendieck topos, which will become the classifying topos for $\mathbb{T}$. There are two natural coverages for geometric categories, which one can show are equivalent.

Proposition 4.1. Let $\mathcal{C}$ be a small geometric category, and let $(f_i : A_i \to B)_{i \in I}$ be a family of morphisms with codomain B in $\mathcal{C}$. Then the following are equivalent.

(i) the images of the f_i cover B, that is, $\bigcup \mathrm{im}\, f_i$ is the maximal subobject;

(ii) the f_i are an *epimorphic family*: if $g, h : B \rightrightarrows C$ are such that $gf_i = hf_i$ for each i, then $g = h$.

The proof is direct. We therefore make the following definition.

Definition 4.2. The *canonical coverage* of a small geometric category $\mathcal{C}$ consists of those sieves that are epimorphic families. It is straightforward to show that this is a coverage.

Recall from remark 1.12 (ii) that the underlying category of any site is embedded in its sheaf topos by the composite **AY**. If J is the canonical coverage, we have a more explicit form of this embedding.

Proposition 4.3. Let $\mathcal{C}$ be a small geometric category, and let J be its canonical coverage. Then every representable presheaf is a J-sheaf. That is, the Yoneda embedding $\mathbf{Y} : \mathcal{C} \to \mathbf{Psh}(\mathcal{C})$ factors through the inclusion functor $\mathbf{I} : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Psh}(\mathcal{C})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\mathbf{Z}"'] \arrow[dr, "\mathbf{Y}"] & \\
\mathbf{Sh}(\mathcal{C}, J) \arrow[r, "\mathbf{I}"'] & \mathbf{Psh}(\mathcal{C})
\end{tikzcd}$$

In particular,

$$\mathbf{AY} \cong \mathbf{Z} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$$

This follows quickly from the definitions. It can also be shown that the canonical coverage is the largest coverage J for which every representable presheaf is a J-sheaf, but we will not use this fact.[9]

We can now build the universal model $U_{\mathbb{T}}$ of $\mathbb{T}$ in the Grothendieck topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$: it is given by the image of $M_{\mathbb{T}}$ under $\mathbf{Z} : \mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$. To show that $U_{\mathbb{T}}$ is a model of $\mathbb{T}$, it suffices to show that $\mathbf{Z}$ is geometric.

Proposition 4.4. For any small geometric category $\mathcal{C}$, the functor $\mathbf{Z} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ is geometric.

Proof. We show that $\mathbf{AY}$ is geometric. Since $\mathbf{A}$ is the inverse image of a geometric morphism, it is a geometric functor, so it suffices to prove that $\mathbf{Y}$ is also a geometric functor. But this is easy, since all relevant geometric constructs are computed pointwise in presheaf categories. ☐

In addition, the sequents satisfied by the universal model are precisely those sequents that $\mathbb{T}$ proves. This follows from the fact that $\mathbf{Z}$ is *conservative* (that is, reflects isomorphisms); in general, conservative geometric functors preserve and reflect satisfaction of geometric sequents.

Corollary 4.5 (completeness in Grothendieck toposes). For every geometric sequent $\varphi \vdash_{\vec{x}} \psi$,

$$(\mathbb{T} \text{ proves } \varphi \vdash_{\vec{x}} \psi) \leftrightarrow (\text{every model of } \mathbb{T} \text{ in a Grothendieck topos satisfies } \varphi \vdash_{\vec{x}} \psi)$$

4.2 Continuous functors

Recall from definition 2.16 that to show that $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$ is the classifying topos for $\mathbb{T}$, we must prove that

$$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathfrak{Top}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J))$$

We have already shown that

$$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathfrak{Geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$$

Our aim in the remainder of this section is to show that the two right-hand sides are equivalent categories. In this section, we will establish the main result that categorises geometric morphisms into such sheaf toposes.

Definition 4.6. Let $(\mathcal{C}, J)$ be a small site where $\mathcal{C}$ has finite limits, and let $\mathcal{E}$ be a Grothendieck topos. We say that a functor $\mathcal{C} \to \mathcal{E}$ is *J-continuous* if it sends J-covering sieves to epimorphic families in $\mathcal{E}$.

---

[9] More generally, the canonical coverage for any small category is typically defined to be the largest coverage for which all representable presheaves are sheaves. As with geometric categories, certain other kinds of categories have alternative equivalent ways with which the canonical coverage may be defined. A discussion of these conditions can be found in [2, pp. 542–544].

Remark 4.7. Many sources (such as [3, p. 116]) use the word *continuous* to describe functors that preserve certain limits. To avoid ambiguity, we will always refer to this behaviour as 'preserving limits' rather than 'continuity'.

Suppose that f is a geometric morphism $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$. We obtain a functor $F : \mathcal{C} \to \mathcal{E}$ given by the composite

$$\mathcal{C} \xrightarrow{\mathbf{Y}} \mathbf{Psh}(\mathcal{C}) \xrightarrow{\mathbf{A}} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{f^\star} \mathcal{E}$$

Lemma 4.8. Let $(\mathcal{C}, J)$ be a small site where $\mathcal{C}$ has finite limits, and let $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ be a geometric morphism. Then the functor $F = f^\star \mathbf{AY}$ is J-continuous.

We will prove this result in the special case where $\mathcal{C}$ is a small geometric category and J is the canonical coverage.

Proof. By proposition 4.1, the epimorphic families in $\mathbf{Sh}(\mathcal{C}, J)$ and $\mathcal{E}$ are precisely those families of morphisms whose images have maximal union. Since $f^\star$ preserves small unions, it suffices to show that $\mathbf{AY}$ sends J-covering sieves to families whose images have maximal union, but this follows directly from the sheaf axiom. □

Additionally, F preserves all finite limits as a composite of functors with this property. It turns out that every J-continuous functor $\mathcal{C} \to \mathcal{E}$ that preserves all finite limits arises in this way, up to natural isomorphism. The mechanics of the proof have little relevance to the rest of this essay, so are skipped for brevity; for full details, see [4, pp. 390–409].

Theorem 4.9. Let $(\mathcal{C}, J)$ be a small site where $\mathcal{C}$ has finite limits. Then for any Grothendieck topos $\mathcal{E}$, there is an equivalence of categories

$$\mathfrak{Top}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

where $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$ is the full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the J-continuous functors that preserve finite limits, and where the forward direction of the equivalence is given by mapping a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the composite

$$\mathcal{C} \xrightarrow{\mathbf{Y}} \mathbf{Psh}(\mathcal{C}) \xrightarrow{\mathbf{A}} \mathbf{Sh}(\mathcal{C}, J) \xrightarrow{f^\star} \mathcal{E}$$

4.3 Existence of classifying toposes

By § 4.2 and theorem 3.14, to prove that $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$ is the classifying topos for $\mathbb{T}$, it suffices to establish an equivalence

$$\mathfrak{Geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E}) \simeq \mathbf{Flat}_J(\mathcal{C}_{\mathbb{T}}, \mathcal{E})$$

We will show that more is true: they are equal as full subcategories of $[\mathcal{C}_{\mathbb{T}}, \mathcal{E}]$.

Lemma 4.10. Let $\mathcal{C}$ be a geometric category, let $\mathcal{E}$ be a Grothendieck topos, and let $F : \mathcal{C} \to \mathcal{E}$ be a functor between them that preserves finite limits. Then F is geometric if and only if it is continuous for the canonical coverage J.

Proof. We have the following sequence of bi-implications.

F is J-continuous	
$\leftrightarrow$ F preserves epimorphic families	by definitions 4.2 and 4.6
$\leftrightarrow$ F preserves epimorphisms and small unions of subobjects	by proposition 4.1
$\leftrightarrow$ F preserves regular epimorphisms and small unions of subobjects	by proposition 3.8
$\leftrightarrow$ F is geometric	by definition 3.6

□

We thus obtain the composite equivalence

$$\mathfrak{Top}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)) \overset{4.9}{\simeq} \mathbf{Flat}_J(\mathcal{C}_{\mathbb{T}}, \mathcal{E}) \overset{4.10}{=} \mathfrak{Geom}(\mathcal{C}_{\mathbb{T}}, \mathcal{E}) \overset{3.14}{\simeq} \mathbb{T}\text{-Mod}(\mathcal{E})$$

as claimed, where a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$ is mapped to the model

$$f^\star(U_{\mathbb{T}}) = f^\star(\mathbf{Z}(M_{\mathbb{T}}))$$

Theorem 4.11. The Grothendieck topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$ is the classifying topos for $\mathbb{T}$, and the universal model is $\mathbf{Z}(M_{\mathbb{T}})$.

Conclusion and further results

We have demonstrated in theorem 4.11 that a classifying topos exists for any geometric theory $\mathbb{T}$, and that it can be constructed using the syntax of $\mathbb{T}$. Every model of $\mathbb{T}$ in a Grothendieck topos $\mathcal{E}$ is of the form $f^\star(U_{\mathbb{T}})$, where f is a geometric morphism $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$, and the universal model $U_{\mathbb{T}}$ is the image of the generic model $M_{\mathbb{T}}$ under the Yoneda embedding $\mathbf{Z} : \mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J)$. Moreover, we have shown in § 4.1 that the geometric sequents satisfied by $U_{\mathbb{T}}$ are precisely those that $\mathbb{T}$ proves, and thus that geometric logic is complete for models in Grothendieck toposes.

If the theory takes a particularly simple form, we can often provide a more explicit description of its classifying topos $\mathbf{Set}[\mathbb{T}]$. Suppose that $\mathbb{T}$ is *algebraic*: it has a single sort and no relation symbols, and all of its axioms are of the form $\top \vdash_{\vec{x}} (s = t)$. Then the classifying topos for $\mathbb{T}$ is equivalent to $\mathbf{Psh}(\mathcal{C}_{\mathbb{T}})$; no coverage is needed. We can also write the classifying topos as $\mathbf{Psh}(\mathbb{T}\text{-Mod}(\mathbf{Set})_\omega{}^{\mathrm{op}})$, where $\mathbb{T}\text{-Mod}(\mathbf{Set})_\omega$ is the category of $\mathbb{T}$-models in $\mathbf{Set}$ that are finitely presented in a suitable sense. This generalises our earlier example of rings from § 2.6, in which we showed that the classifying topos for rings was $\mathbf{Psh}(\mathbf{FpRing}^{\mathrm{op}})$. [2, p. 891]

It can be shown that every Grothendieck topos $\mathcal{E}$ is the classifying topos of some theory. First, we choose a site $(\mathcal{C}, J)$ for $\mathcal{E}$ where $\mathcal{C}$ has finite limits; this can always be done by theorem C2.2.8 of [2, p. 553]. Let Σ be the signature containing one sort $\ulcorner A \urcorner$ for each object A of $\mathcal{C}$, and one function symbol $\ulcorner f \urcorner : [\ulcorner A \urcorner] \to \ulcorner B \urcorner$ for each morphism $f : A \to B$ of $\mathcal{C}$. We can easily produce a theory $\mathbb{T}$ over Σ whose models in a geometric category $\mathcal{D}$ correspond to functors $\mathcal{C} \to \mathcal{D}$ that preserve finite limits. The details can be found in [2, pp. 837, 846]. For each covering family $(f_i : B_i \to A)_{i \in I}$ of J, we add the sequents

$$\top \vdash_{[x:A]} \bigvee_{i \in I} (\exists y_i : B_i.\, f_i(y_i) = x)$$

This produces a theory $\mathbb{T}'$ whose models in a Grothendieck topos $\mathcal{F}$ correspond to J-continuous functors $\mathcal{C} \to \mathcal{F}$ that preserve finite limits. Then $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ is the classifying topos for $\mathbb{T}'$ by theorem 4.9. [2, p. 897]

In §§ 1.5 and 1.6, we showed that Grothendieck toposes have a subobject classifier and power objects. There is a more general definition of *(elementary) topos*: a topos is a category $\mathcal{E}$ with all finite limits, a subobject classifier, and power objects. [4, p. 161] With this definition, §§ 1.5 and 1.6 can be viewed as proofs that Grothendieck toposes are elementary toposes. It can be shown that toposes have exponential objects, finite colimits, and image factorisations; see § IV of [4] for details. These constructions allow us to interpret higher order logic in any topos, an idea that is explored in § D4 of [2]. In the more general setting of arbitrary toposes, $\mathbf{Set}[\mathbb{T}]$ is called a classifying topos if there is an equivalence

$$\mathbb{T}\text{-Mod}(\mathcal{E}) \simeq \mathfrak{Top}/\mathbf{Set}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$$

for every topos that admits a geometric morphism to **Set**, where $\mathfrak{Top}/\mathbf{Set}$ is the 2-category of toposes over **Set**. This is the approach used in [2, p. 890]. Note that, as was the case with Grothendieck toposes, elementary toposes admit at most one geometric morphism to **Set**.

Notation

$*$	example 1.2 (iii)	the one-point topological space $\{\bullet\}$
$[\mathcal{C}, \mathcal{D}]$		the category of functors $\mathcal{C} \to \mathcal{D}$, and the natural transformations between them
$[\![\vec{x}\,.\,t]\!]_M$	definition 2.7	the interpretation of a term-in-context $\vec{x}\,.\,t$ in a model M
$[\![\vec{x}\,.\,\varphi]\!]_M$	definition 2.10	the interpretation of a formula-in-context $\vec{x}\,.\,\varphi$ in a model M
$\{\vec{x}\,.\,\varphi\}$	definition 3.2	the object in a syntactic category corresponding to the formula-in-context $\vec{x}\,.\,\varphi$
$\mathbf{A}$	remark 1.9 (iii)	the associated sheaf functor
$\mathcal{C}_{\mathbb{T}}$	definition 3.2	the syntactic category of a geometric theory $\mathbb{T}$
$\mathrm{Colim}\, D$		the colimit cocone of the diagram D
$\mathfrak{Geom}$	definition 3.6	the 2-category of geometric categories, geometric functors, and the natural transformations between them
$\mathbf{Flat}(\mathcal{C}, \mathcal{E})$	§ 2.6	the full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the functors that preserve finite limits
$\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$	theorem 4.9	the full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the J-continuous functors that preserve finite limits
$\mathbf{I}$	remark 1.9 (iii)	the inclusion functor
im	§ 1.7	the image of the morphism f
$\mathrm{Lim}\, D$		the limit cone of the diagram D
$M_{\mathbb{T}}$	definition 3.9	the generic model of a geometric theory $\mathbb{T}$
$\mathbb{T}\text{-Mod}(\mathcal{E})$	definition 2.14	the category of $\mathbb{T}$-models in $\mathcal{E}$
$\mathcal{O}(X)$	remark 1.3 (i)	the poset of open sets of the topological space X, ordered by inclusion
$(-)^{\mathrm{op}}$		the opposite category

P	remark 1.19 (i)	the power object functor $\Omega^{(-)} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$
$\mathbf{Psh}(\mathcal{C})$	remark 1.9 (ii)	the category of presheaves $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ on the category $\mathcal{C}$, and the natural transformations between them
res^{U}_{V}	definition 1.1	the restriction maps for sheaves on topological spaces
$\mathbf{Set}$		the category of sets and functions
$\mathbf{Set}[\mathbb{T}]$	definition 2.16	the classifying topos of $\mathbb{T}$
$\mathbf{Sh}(X)$	definition 1.1	the category of sheaves on the topological space X
$\mathbf{Sh}(\mathcal{C}, J)$	definition 1.8	the category of sheaves on the site $(\mathcal{C}, J)$
$\Sigma\text{-}\mathrm{Str}(\mathcal{E})$	definition 2.3	the category of Σ-structures in $\mathcal{E}$
Sub	§ 1.6	the subobject functor
$\mathfrak{Top}$	definition 1.14	the 2-category of Grothendieck toposes, geometric morphisms, and the natural transformations between them
$U_{\mathbb{T}}$	§ 2.5	the universal model of a geometric theory $\mathbb{T}$
$\mathbf{Y}$	remark 1.12 (ii)	the Yoneda embedding
$\mathbf{Z}$	proposition 4.3	the factorisation of Yoneda embedding $\mathbf{Y}$ through the inclusion functor $\mathbf{I}$
Ω	definition 1.17	the subobject classifier

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