Classifying Topoi

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This essay is about topoi, a certain kind of structured category invented by Grothendieck in the early 1960s. Grothendieck was working towards the Weil conjectures and to do this needed to construct a new cohomology theory analogous to sheaf cohomology but with a more general notion of open cover. Category Theory allowed him to extend the definition of covering from topological spaces to covering objects in general categories via the concept called a *site* which induces a *Grothendieck topos*. Subsequent to his work, a number of researchers in the 1970s developed the field in a completely different direction when it was found that these categories could be used to approach mathematical logic in a new way that generalised the classical approach of building models of theories out of sets.

In this essay, we hope to give some idea of the field of categorical logic with two main targets: the existence of classifying topoi for geometric theories and then additionally internal language of topoi. We hope to emphasise the inherent surprise that comes from the fact that the objects of study were initially defined for geometric purposes and yet have a wholly different logical side to them.

- In Section 1, we will introduce **sites**, a way of giving a category $\mathcal{C}$ a topological flavour which is precisely what's needed to define covers of objects and then sheaves on the category. This yields the concept of a **Grothendieck topos**, which is the category of sheaves we obtain from $\mathcal{C}$, and **geometric morphisms** that go between these.
- In Section 2, we introduce **elementary topoi**, a more general class of structured category defined by Tierney and Lawvere, and show that despite the rather modest list of requirements in the definition they have a variety of properties which set them apart from general categories and in particular make them act much like Set.
- In Section 3, we compare the two notions of topos, showing that all Grothendieck topoi are elementary and have more structure besides as, for example, they have all small colimits while some examples of elementary topoi only have finite colimits.
- Section 4 finally introduces the fundamentals of categorical logic, explaining how higher order (intuitionistic) languages and theories can be given interpretations, models and homomorphisms in topoi in a way that extends the classical way of working with sets. We illustrate this with a geometric example, the theory of local rings, and a logical example, the *internal language of a topos*. We observe that the assignment which takes a topos $\mathcal{E}$ to its category of models $\mathrm{Mod}(T, \mathcal{E})$ of a theory T isn't in general functorial but becomes so if we restrict ourselves to a certain fragment of first-order logic dubbed **geometric logic**.
- Section 5 produces the most important proof of the essay: that for any geometric theory T, the contravariant functor $\mathcal{E} \to \mathrm{Mod}(T, \mathcal{E})$ is represented by its so-called **classifying topos** and Section 6 constructs two of these directly.
- Finally, we felt it would be a great shame if the reader was not given at least a brief discussion of the internal language of a topos so in Section 7 we state Joyal's *sheaf semantics*, which allows us to work it more easily, and then give an example of its use to prove a simple but nontrivial generalisation of the Weierstrass Approximation Theorem from real analysis which we hope the reader will find as shocking as the author did.

The main source for this essay has been [12], which provides for the most part a satisfactory and self-contained account of this theory. However, the proofs in Sections 2,3 are inspired by the concise presentation given in [10]. The presentation of the logic given in Section 4 was guided by the foundational reference [11] as well as the PhD thesis [1]. The proof given of the existence of classifying topoi is based on that in [12] but the brief explanation of Diaconescu's theorem was guided by the discussion in [5], §1.1.5. The final section was inspired by the work done in [6] and [2] but the details and example are down to the author.

1 Sites

Grothendieck was interested in taking the theory of sheaves on topological spaces and generalising it so that sheaves could now be defined not only on the category of open subsets of a space but on examples such as Sch, the category of all schemes. To understand this, we recall the classical case.

Definition 1.1. Let X be a topological space and $\mathcal{O}$ its poset of open subsets, viewed as a category. A **presheaf** is a functor $P : \mathcal{O}^{\mathrm{op}} \to \mathrm{Set}$. If $U \subseteq X$ is open, $x \in P(U)$ and $\iota : V \to U$ is an inclusion of open subsets then we notate $x|_\iota = x|_V = P(\iota)(x) \in P(V)$ for the **restriction of** x **to** V or **along** ι.

If $\mathcal{U}$ is an open cover of U in X then $\{x_V\}_{V\in\mathcal{U}}$ is a **matching family** for U, P iff

1. for each $V \in \mathcal{U}$, $x_V \in P(V)$
2. for each $V, W \in \mathcal{U}$ we have $x_V|_{W\cap V} = x_W|_{W\cap V}$.

Then P is a **sheaf** (of sets) if for any matching family $\{x_V\}$ as above there exists a unique $x_U \in P(U)$ so that $x_U|_V = x_V$ for all $V \in \mathcal{U}$.

Morphisms between sheaves are just natural transformations between the two functors and this allows us to make presheaves into a category $[\mathcal{O}^{\mathrm{op}}, \mathrm{Set}]$ and similarly we can define the full subcategory $\mathrm{Sh}(X)$ of sheaves. In order to mimic this construction on general categories $\mathcal{C}$, it is clear that we can just define presheaves to more generally refer to contravariant functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and the category of presheaves is just the functor category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. However, to define the *sheaves* among these we have to think more about the above situation.

The above definition didn't actually make use of much of the structure of the topology: we made reference to inclusions of opens $i : U \to V$, which can be replaced with morphisms $f : C \to D$ between objects of our category $\mathcal{C}$, and to *open covers*.

Definition 1.2. Let X be a topological space. A **sieve** $\mathcal{S}$ on an open subset $U \subseteq X$ is a collection of open subsets of U closed under taking open subsets, i.e. if $V \in \mathcal{S}$ and $W \subseteq V$ is open then $W \in \mathcal{S}$. A **covering sieve** is a sieve on U such that $\bigcup \mathcal{S} = U$.

More generally, if $\mathcal{C}$ is a category then a **sieve** $\mathcal{S}$ on an object C of $\mathcal{C}$ is a collection of arrows with codomain C such that if $f : B \to C$ is in $\mathcal{S}$ and $g : A \to B$ is an arrow in $\mathcal{C}$ then $f \circ g : A \to C$ is also in $\mathcal{S}$.

Notice that the two notions of sieve agree when you take $\mathcal{C}$ to be the category $\mathcal{O}$ for X and without loss of generality when working with sheaves we may assume that any open cover is in particular a sieve since any open cover $\mathcal{U}$ generates a unique sieve by closing under taking open subsets and similarly any matching family $\{x_V\}_{V\in\mathcal{U}}$ extends in a unique way to this sieve by taking restrictions of pre-existing elements x_V to the new subsets $W \subseteq V$ so that $x_W := x_V|_W$. This construction is well-defined by axiom **(2)** of matching families.

What properties do sieves have in the topological context?

1. If $U \subseteq X$ is an open then the sieve of *all* open subsets of U (the **maximal sieve**) is a covering sieve.
2. If $\mathcal{S}$ is a sieve on U and $\iota : V \to U$ is an inclusion of opens we can consider the *pullback sieve* $\iota^*\mathcal{S} := \{W \cap V \ : \ W \in S\}$ or equivalently it is the set of open subsets of U in $\mathcal{S}$ which factor through ι; this forms a sieve on V. Clearly if $\mathcal{S}$ covers U then $\iota^* S$ covers V.
3. If $\mathcal{S}$ is a covering sieve on U and $\mathcal{S}'$ is a sieve on U such that $\mathcal{S}'$ covers every $V \in \mathcal{S}$, then $\mathcal{S}'$ covers U. Phrased in the above language, if $\iota^*\mathcal{S}'$ is a covering sieve for every $\iota : V \to U$ in $\mathcal{S}$ then $\mathcal{S}'$ is a covering sieve.

Grothendieck's insight was that these observations were sufficient to build a good notion of coverings and thus sheaves on any category and indeed, having noticed the above, the below definitions almost write themselves:

Definition 1.3. Let $\mathcal{C}$ be a category. A **Grothendieck topology** $\mathcal{J}$ on $\mathcal{C}$ is an assignment for each object C of $\mathcal{C}$ a collection $\mathcal{J}(C)$ of sieves, called the **covering sieves**, on C such that following axioms are satisfied:

1. The maximal sieve is in $\mathcal{J}(C)$ for all C.
2. If $\mathcal{S}$ is a sieve in $\mathcal{J}(C)$ and $f : D \to C$ is a morphism we have that the *pullback sieve* $f^*\mathcal{S} := \{g : E \to D \ : \ f \circ g \in \mathcal{S}\}$ is in $\mathcal{J}(D)$.
3. If $\mathcal{S}$ is a sieve in $\mathcal{J}(C)$ and $\mathcal{S}'$ is a sieve on C such that $f^*\mathcal{S}'$ is in $\mathcal{J}(D)$ for all $f : D \to C$ in $\mathcal{S}$ then $\mathcal{S}'$ is in $\mathcal{J}(C)$.

A pair $(\mathcal{C}, \mathcal{J})$ of a category along with a specified topology is called a **site**. If $P : \mathcal{C}^{\text{op}} \to \text{Set}$ is a presheaf, a **matching family** for P and a sieve $\mathcal{S} \in \mathcal{J}(C)$ is a collection of elements $\{x_f\}_{f:D\to C\in\mathcal{S}}$ iff

1. for each $f : D \to C$ in $\mathcal{S}$, $x_f \in P(D)$
2. for each $f : D \to C, g : E \to D$, we have $x_f|_g = x_{fg}$.

Then P is a **sheaf** if for any matching family $\{x_f\}$ as above there exists a unique $x_C = x_{\text{id}:C\to C} \in P(C)$ so that $x_C|_f = x_f$ for all $f : D \to C$ in $\mathcal{S}$.

We then write $\text{Sh}(\mathcal{C}, \mathcal{J})$ for the full subcategory of $[\mathcal{C}^{\text{op}}, \text{Set}]$ consisting of the sheaves with respect to $\mathcal{J}$ and any category which is equivalent to the category of sheaves on a site in this way is called a **Grothendieck topos**.

Naturally for any large definition like this it is indispensable to have a good stock of examples.

Example 1.1 (Topological)**.** Given Grothendieck's observations above about covering sieves in topological spaces, the most obvious example of a site comes from topology. Given a topological space X, we define a (Grothendieck) topology $\mathcal{J}$ on its category of open sets $\mathcal{O}$ where an open U is covered by a sieve of opens $\mathcal{S}$ iff the union of their included images is all of U (i.e. the sieve is an open cover). Then $\text{Sh}(\mathcal{O}, \mathcal{J})$ is exactly the normal category of sheaves of sets on that space.

Example 1.2 (Trivial)**.** Perhaps even more simply, any category $\mathcal{C}$ can be given the **trivial topology** in which only the maximal sieve on each object is declared to be a covering sieve. In this case, the category of sheaves on the site is simply the presheaf topos $[\mathcal{C}^{\text{op}}, \text{Set}]$.

It is frequently convenient to describe a Grothendieck topology as being generated by a simpler structure which is sometimes called a **basis** for such a topology, though it is more accurate to say that it corresponds to axiomatising open covers on a category instead of covering sieves like above:

Definition 1.4 (Basis)**.** Let $\mathcal{C}$ be a category with pullbacks. A **basis** is an assignment K of a collection of families of morphisms or *covers* for each object C such that:

1. for any C, the family $\{\text{id} : C \to C\}$ covers C,
2. for any cover $\{f_i : B_i \to C\}$ of C and morphism $g : D \to C$ the collection of pullbacks $\{\pi_D : B_i \times_C D \to D\}$ along g is a cover,
3. for any cover $\{f_i : B_i \to C\}$ and any collection of covers $\{g_{ij} : D_j \to B_i\}$ (one for each i), the collection of composites $\{f_i \circ g_{ij} : D_j \to C\}$ is a cover.

The interested reader may check that the axioms respectively correspond to the facts that in a topological space every open covers itself, every open cover of an open U covers every subset $V \subseteq U$ and a collection of opens which covers each member of an open cover of U is itself a cover of U. Much as how we said above that open covers relate to sieves of opens in the topological setting, we have:

Proposition 1.1

Given a category $\mathcal{C}$ with pullbacks and a basis K on it, the sieves on objects C which contain a member of $K(C)$ together form a Grothendieck topology $\mathcal{J}$ on $\mathcal{C}$.

Proof:
We simply verify the axioms for a topology given above. For any object C, any sieve $\mathcal{S}$ which contains $\mathrm{id} : C \to C$ must contain $g = \mathrm{id} \circ g : D \to C$ for any arrow $g : D \to C$ and thus axiom **(1)** of a basis means $\mathcal{J}$ contains the maximal sieve on every object.

Next if $\mathcal{S}$ is a sieve on C in $\mathcal{J}$ then it by definition contains some cover $\{f_i : B_i \to C\}$ in $K(C)$. But then for any $g : D \to C$ we know that

$$\begin{array}{ccc} B_i \times_C D & \xrightarrow{\pi_{B_i}} & B_i \\ {\scriptstyle \pi_D}\downarrow & & \downarrow{\scriptstyle f_i} \\ D & \xrightarrow{g} & C \end{array}$$

commutes so $g \circ \pi_D = f_i \circ \pi_{B_i} \in \mathcal{S}$ so $\pi_D : B_i \times_C D \to D$ is in $g^*\mathcal{S}$ for each i. Thus $g^*\mathcal{S}$ is in $\mathcal{J}$, verifying axiom **(2)** of a topology.

Finally, suppose $\mathcal{S}$ is a sieve in $\mathcal{J}(C)$ and thus contains some cover $\{f_i : B_i \to C\}$ as before and $\mathcal{S}'$ is a sieve on C such that $f^*\mathcal{S}'$ is in $\mathcal{J}(D)$ for all $f : D \to C$ in $\mathcal{S}$. In particular, for each of the f_i we can look at $f_i^*\mathcal{S}'$ and see that it must contain a member of $K(B_i)$, i.e. there is some cover $\{g_{ij} : B_{ij} \to B_i\}$ so that each composition $f_i \circ g_{ij} : B_{ij} \to C$ is in $\mathcal{S}'$. But axiom **(3)** of a basis tells us that the collection of compositions $\{f_i \circ g_{ij} : B_{ij} \to C\}$ for all i, j is in $K(C)$ and since $\mathcal{S}'$ contains it, $\mathcal{S}'$ is a covering sieve. This shows $\mathcal{J}$ satisfies axiom **(3)** of a topology and we are done. □

This allows us to give a few more relevant examples of Grothendieck topoi.

Example 1.3. The **big Zariski site** is the category $\mathrm{Aff} \cong \mathrm{CRing}^{\mathrm{op}}$ of affine schemes and scheme morphisms with a basis where covers are essentially the same notion as open covers of sets in the affine topology in $\operatorname{Spec} R$ for a ring R. In particular, for an object $\operatorname{Spec} R$ of Aff we say that a collection of morphisms $f_i : \operatorname{Spec} A_i \to \operatorname{Spec} R$ is a cover iff each $A_i = R_{f_i}$ is the localisation of R at some element f_i, the maps $f_i : \operatorname{Spec} R_{f_i} \to \operatorname{Spec} R$ are the canonical inclusions which are dual to the localisation map $R \to R_{f_i}$ and the distinguished opens $U_{f_i} \subseteq \operatorname{Spec} R_i$, which are the images of these maps, cover $\operatorname{Spec} R_i$.

Writing this in terms of viewing the category as $\mathrm{CRing}^{\mathrm{op}}$, covers of an object R are collections of localisation maps $R \to R_{f_i}$ such that the ideal generated by the elements is all of R: $\langle f_1, \dots, f_n \rangle = R$. Let's check this does form a basis:

- Clearly for any ring R, $\mathrm{id} : \operatorname{Spec} R \to \operatorname{Spec} R$ covers $\operatorname{Spec} R$.
- If $\operatorname{Spec} R$ is covered by the localisations $\{R \to R_{f_i}\}$ and $\varphi : R \to S$ is a ring homomorphism then a pullback $\pi_{\operatorname{Spec} S} : \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} R_{f_i} \to \operatorname{Spec} S$ in $\mathrm{CRing}^{\mathrm{op}}$ corresponds to the righthand vertical "inclusion" map in the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_{f_i} & \longrightarrow & S \otimes_R R_{f_i} \end{array}$$

which is a pushout but $S \otimes_R R_{f_i} \cong S_{\varphi(f_i)}$ by viewing S as an R-algebra and the tensor product as an extension of scalars. Thus the localisation maps $\{R \to R_{f_i}\}$ pullback to localisation maps $\{S \to S_{\varphi(f_i)}\}$ and

$$1_R = \sum_i a_i f_i \implies 1_S = \varphi(1_R) = \sum_i \varphi(a_i)\varphi(f_i)$$

so since the initial localisations covered Spec R the pullbacks cover Spec S.

- For the final composition requirement, notice that if $R \to R_f, R_f \to (R_f)_{\tilde{g}}$ are localisation maps then, writing $\tilde{g} = \frac{g}{f^n}$ for some g in the image of $R \to R_f$, the composition is up to isomorphism the localisation map $R \to R_{fg}$. Thus if $\{R \to R_{f_i}\}$ are a cover and each of the localisations have covers $\{R_{f_i} \to (R_{f_i})_{\tilde{g_{ij}}}\}$ we can compose them and get a family of localisation maps $\{R \to R_{f_i g_{ij}}\}$ which clearly form a cover.

Example 1.4. The **big étale site** is the category of schemes Sch with scheme morphisms with a basis given by collections of flat unramified maps $\{f : X_i \to X\}$, i.e. maps such that

- for all points $p \in X_i$ the stalk map $f_p^\sharp : \mathcal{O}_{X,f(p)} \to \mathcal{O}_{X_i,p}$ is flat,
- the residue field extension $k(p)/k(f(p))$ is separable,
- $f^\sharp(\mathfrak{m}_{f(p)}) = \mathfrak{m}_p$ for $\mathfrak{m}_x \subseteq \mathcal{O}_{X_i,p}, \mathfrak{m}_{f(p)} \subseteq \mathcal{O}_{X,f(p)}$ the unique maximal ideals,
- they are locally of finite presentation,

such that their images union to give X. We won't check this is a basis but simply remark that to begin the theory of étale cohomology we might consider the induced site on the slice category Sch$/X$ for a scheme morphism and compute the resulting sheaf cohomology.

Example 1.5. Let $\mathcal{C}$ be a category finite limits, finite colimits and all exponential objects such that pullbacks preserve epimorphisms and coproducts. Then setting covers of objects to be finite families of morphisms $f_i : C_i \to C$ such that their coproduct map $f : \coprod_{i=1}^n C_i \to C$ is epimorphic forms a basis.

This is easily checked but is also discussed in [12] Chapter X, §3.

This example will be useful later in §5 where we will use it to construct an important site. As a final comment on sites, notice that the condition that a presheaf has a unique amalgamation for every matching family $(x_f)_f$ with a covering sieve $\mathcal{S}$ on an object C can be phrased as $P(C)$ being an equaliser (just as for sheaves on topological spaces)

$$P(C) \xrightarrow{\ e\ } \prod_{(f:D\to C)\in\mathcal{S}} P(D) \underset{p}{\overset{m}{\rightrightarrows}} \prod_{(f:D\to C)\in\mathcal{S},g:E\to D} P(E)$$

where $e(x) = (P(f)(x))_f$ sends x to its associated matching family on $\mathcal{S}$, $m((x_f)_f) = (x_{fg})_{f,g}$ sends a given component x_f to the member x_{fg} of the tuple $(x_f)_f$ in the (f,g)-component of the second product and $p((x_f)_f) = (P(g)(x_f))_{f,g}$ sends x_f to its restriction $P(g)(x_f)$ in the (f,g)-component. We will use this later in §3 to show that limits of diagrams of sheaves are also sheaves.

Any new type of object deserves a notion of morphism and so we define:

Definition 1.5 (Geometric Morphism)**.** Let $\mathcal{E}, \mathcal{F}$ be Grothendieck topoi. A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ is an adjunction

$$\mathcal{E} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{F} \quad (f^* \dashv f_*)$$

such that the left adjoint $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits. The left adjoint f^* is called the **inverse image** of the geometric morphism while f_* is called the **direct image**. A **natural transformation** between two geometric morphisms is defined to be a natural transformation between their inverse images and the category of geometric morphisms $\mathcal{E} \to \mathcal{F}$ and natural transformations between them is denoted $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})$.

Notice that if $f, g : \mathcal{E} \to \mathcal{F}$ are geometric morphisms then a natural transformation $\alpha : f^* \to g^*$ between their inverse images induces a natural transformation $\overline{\alpha} : g_* \to f_*$ between their direct images because the adjunctions give a commutative square

$$\begin{array}{ccc} \mathcal{E}(f^*(F), E) & \xrightarrow{\ \cong\ } & \mathcal{F}(F, f_*(E)) \\ \Big\uparrow{\scriptstyle \alpha_F^*} & & \Big\uparrow{\scriptstyle (\beta_E)_*} \\ \mathcal{E}(g^*(F), E) & \xrightarrow{\ \cong\ } & \mathcal{F}(F, g_*(E)) \end{array}$$

so really a natural transformation between geometric morphisms is a pair of transformations between their direct and inverse images but in order to make sure these correspond to each along the adjunctions as above it's simplest to just define it on the inverse images.

The definition of a geometric morphism may seem quite unmotivated at first but the intuition is once again spatial, so let's look at the motivating example.

Example 1.6. Let $f : X \to Y$ be a continuous map between topological spaces X, Y. As will be familiar to those with experience in algebraic geometry, any sheaf $\mathcal{S}$ on X can then be turned into a sheaf $f_*\mathcal{S}$ on Y (the "direct image sheaf") by defining $(f_*\mathcal{S})(U) := \mathcal{S}(f^{-1}(U))$ and this process is functorial in an obvious way. Similarly any sheaf $\mathcal{R}$ on Y can be turned into $f^*\mathcal{R}$ on X (the "inverse image sheaf") by defining the presheaf

$$(f^*\mathcal{R})^{\text{pre}}(U) = \{(V, S_V) \ : \ V \text{ open containing } f(U),\ S_V \in \mathcal{R}(V)\}/\!\sim$$

for U open in X and where $\sim$ identifies pairs (V, S_V) which restrict to the same thing on a smaller open containing $f(U)$, i.e. it is the filtered colimit of the sets $\mathcal{R}(V)$ for $V \supseteq f(U)$. Then $f^*\mathcal{R}$ is the sheafification of $(f^*\mathcal{R})^{\text{pre}}$. Again, this is functorial in a clear way.

Thus $f : X \to Y$ induces functors $f_* : \text{Sh}(X) \to \text{Sh}(Y), f^* : \text{Sh}(Y) \to \text{Sh}(X)$ and in algebraic geometry it is further proven that $f^* \dashv f_*$. Furthermore, as proven in say [15], a standard categorical fact is that filtered colimits commute with finite limits in Set so f^* preserves finite limits.

Thus Grothendieck topoi form a category. We will see that the adjoints in the definition of geometric morphism are useful given that Grothendieck topoi have all small limits and colimits

Given Grothendieck topoi are quite spatial in nature and their morphisms are analogous to continuous maps between topological spaces, we might draw a very weak analogy with algebraic topology. Algebraic topology abounds with important functors, such as the generalised homology and cohomology theories, and in 1962 Edgar Brown [4] achieved an important characterisation of those which are representable:

Theorem 1.1 (*Brown Representability*)

Let Htpy_* refer to the category of based topological spaces (X, x) with morphisms being based homotopy classes of based continuous functions $f : (X, x) \to (Y, y)$, i.e. those such that $f(x) = y$. Then a functor $F : \text{Htpy}_* \to \text{Set}_*$ is representable iff

- it takes coproducts to products,
- it takes weak pushouts to weak pullbacks.

"Weak ..." refers to objects which satisfy the existence part of the definition of the (co)limit but not the uniqueness, i.e. there exists a map making the relevant triangles commute from them to any cone above/below the appropriate diagram. We might wonder if there is a nice characterisation or at least sufficient condition for functors on the category of Grothendieck topoi to be representable... §4's discussion of classifying topoi will answer this in a rather unexpected way!

2 Elementary Topoi

We have discussed some geometry and now it is time for logic. In the early 1970s, Bill Lawvere and Myles Tierney were part of a group who had noticed that Grothendieck topoi are full of great structure, as we will see, and that this structure is such that they look in many ways like the category of sets. Taking this observation further, they realised that they could generalise the notion of $\mathcal{L}$-structures in logic to give interpretations of higher order intuitionistic logic in these categories, as will be discussed in the next section. However, for their purposes the geometry was unnecessary and so they formulated a more general notion of **elementary topos**.

Definition 2.1. An **elementary topos** is a category $\mathcal{C}$ which has limits for all finite diagrams, has all exponential objects and has a **subobject classifier**, i.e. there is an object Ω and a specified arrow true : $1 \to \Omega$ such that for any object C and subobject $C' \rightarrowtail C$ there is a unique bottom horizontal morphism making the diagram

$$\begin{array}{ccc} C' & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ C & \overset{\exists!}{\dashrightarrow} & \Omega \end{array}$$

a pullback.

This section serves as a very brief introduction to the theory of elementary topoi; we will establish enough to discuss interpretations in these categories in §3. Proofs in this section are inspired by those in [10] Chapter 1 and [12] Chapter IV. First, let's ponder the definition: the novel aspect for most readers will be the subobject classifier.

Example 2.1. The motivation for this definition comes from the fact that subsets of a given set S can be equivalently thought of as functions $S \to \{0, 1\}$, i.e. their characteristic functions. In particular, given a subset S' the fact that the square

$$\begin{array}{ccc} S' & \longrightarrow & \{*\} \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ S & \xrightarrow{\chi} & \{0,1\} \end{array}$$

commutes, where true : $\{*\} \to \{0, 1\}$ sends $* \mapsto 1$, tells you that the preimage $\chi^{-1}(1)$ contains S' and then the fact that this square is a pullback tells you that $S' = \chi^{-1}(1)$, i.e. χ is the characteristic function of S'. Thus $\{0, 1\}$ with the arrow true make up a subobject classifier for Set.

Example 2.2. Various other categories which are not topoi can still have subobject classifiers. Top rather famously fails to have many exponential objects but its subobject classifier is the *Sierpinski space* $\mathcal{S}$ whose underlying set is $\{0, 1\}$ and whose topology makes $\{1\}$ open but not $\{0\}$. This is a subobject classifier for just the same reason as $\{0, 1\}$ is in Set except that preimages of $\{1\}$ correspond bijectively with *open* subsets of a given space rather than *all* subsets.

Example 2.3. The category Group is a non-example, i.e. there is no subobject classifier in this category. For suppose $\mathfrak{G}$ was a subobject classifier and the arrow true : $\{e\} \to \mathfrak{G}$ was the only possible group homomorphism. Then any group G has the trivial subgroup so there is a unique homomorphism $\varphi : G \to \mathfrak{G}$ making the diagram

$$\begin{array}{ccc} \{e\} & \longrightarrow & \{e\} \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ G & \overset{\varphi}{\dashrightarrow} & \mathfrak{G} \end{array}$$

a pullback. This precisely means that $\ker \varphi$ is trivial, i.e. φ is injective. But then this gives an immediate contradiction if we consider a group G whose cardinality exceeds that of $\mathfrak{G}$. Similar arguments can be repeated in Ab, R-Mod and various other categories of algebraic structures.

Another way of thinking about the subobject classifier of an elementary topos $\mathcal{E}$ is in terms of the functor $\mathrm{Sub}_{\mathcal{E}} : \mathcal{E} \to \mathrm{Set}$ (assuming here that $\mathcal{E}$ is well-powered), which is defined on objects by outputting the poset of subobjects and is defined on morphisms $f : A \to B$ by considering the *pullback functor* $f^{-1} : \mathrm{Sub}_{\mathcal{E}}(B) \to \mathrm{Sub}_{\mathcal{E}}(A)$ which takes a subobject $B' \rightarrowtail B$ and forms the pullback

$$\begin{array}{ccc} f^{-1}(B') & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

This is well-defined since the pullback of a monomorphism is mono and is functorial since clearly pulling back along $\mathrm{id} : B \to B$ does nothing and pullback squares

$$\begin{array}{ccccc} f^{-1}(g^{-1}(C')) & \longrightarrow & g^{-1}(C') & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array}$$

concatenate in the sense that the outer rectangle is a pullback so that $\mathrm{Sub}_{\mathcal{E}}(g \circ f) = \mathrm{Sub}_{\mathcal{E}}(f) \circ \mathrm{Sub}_{\mathcal{E}}(g)$. Then it is not hard to check that ([12] Chapter I):

Proposition 2.1

The subobject classifier is a representing object for $\mathrm{Sub}_{\mathcal{E}}$, i.e. there is a natural isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(-) \cong \mathrm{Hom}(-, \Omega)$$

Yet another way to think about subobjects in $\mathcal{E}$ is to notice that subobjects of C correspond to morphisms $C \cong 1 \times C \to \Omega$ which have exponential transposes $1 \to \Omega^C$. This naturally suggests focusing on the **powerset functor** $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ which sends objects C to Ω^C and morphisms $f : C \to D$ to the exponential transpose $Pf : \Omega^D \to \Omega^C$ of

$$\Omega^D \times C \xrightarrow{1 \times f} \Omega^D \times D \xrightarrow{\mathrm{ev}} \Omega$$

where $\mathrm{ev} : \Omega^D \times D \to \Omega$ is the evaluation map which comes from transposing $\mathrm{id} : \Omega^D \to \Omega^D$. The powerset functor is interesting especially because it is the key to the first result showing elementary topoi have more structure than their definition suggests.

Theorem 2.1

Elementary topoi have all finite colimits. Moreover, whatever infinite limits they have, they must also have the corresponding colimits.

Proof:
We only sketch the ingenious proof of this, due to Paré [14]. The central observation is that P builds a strong duality between $\mathcal{E}^{\text{op}}$ and $\mathcal{E}$: it has a left adjoint and indeed is monadic.

Lemma 2.1

If we write $P^{\text{op}} : \mathcal{E} \to \mathcal{E}^{\text{op}}$ then $P^{\text{op}} \dashv P$.

Proof:
This is follows from the bijections

$$\text{Hom}_{\mathcal{E}^{\text{op}}}(P^{\text{op}}X, Y) \cong \text{Hom}_{\mathcal{E}}(Y, \Omega^X) \cong \text{Hom}_{\mathcal{E}}(X \times Y, \Omega) \cong \text{Hom}_{\mathcal{E}}(Y \times X, \Omega) \cong \text{Hom}_{\mathcal{E}}(X, PY)$$

which are natural in X, Y. □

To prove P is in fact monadic, we apply what Johnstone called the *Crude Monadicity Theorem* which tells that it's sufficient for P to have a left adjoint, to reflect isomorphisms and for $\mathcal{E}^{\text{op}}$ to have and P to reflect reflexive coequalisers. We know it has the left adjoint and $\mathcal{E}^{\text{op}}$ has all reflexive coequalisers as they are just coreflexive equalisers in $\mathcal{E}$ (which has all finite limits) and the remaining calculations are not difficult[a].

But then $\mathcal{E}^{\text{op}}$ is equivalent to the category of algebras over $\mathcal{E}$ induced by the adjunction and the forgetful functor $\mathcal{E}^{\text{op}} \to \mathcal{E}$ for these algebras creates all limits which exist in $\mathcal{E}$. But limits in $\mathcal{E}^{\text{op}}$ are colimits in $\mathcal{E}$ and so we have proven the theorem! □

[a] Though reflection of isomorphisms requires knowing $\mathcal{E}$ is balanced, which we shall show later in this section.

We now turn to a technical tool McLarty [13] Chapter 17 refers to as the *fundamental theorem of topoi*:

Theorem 2.2

Let $\mathcal{E}$ be an elementary topos and A be an object in $\mathcal{E}$. Then $\mathcal{E}/A$ is a topos and the pullback functor $A^* : \mathcal{E} \to \mathcal{E}/A$ which sends an object $C \mapsto (C \times A \xrightarrow{\pi_A} A)$ and

$$(C \xrightarrow{f} D) \quad \mapsto \quad \begin{array}{ccc} (C \times A & \xrightarrow{f \times \text{id}} & D \times A) \\ & {\scriptstyle \pi_A} \searrow \quad \swarrow {\scriptstyle \pi_A} & \\ & A & \end{array}$$

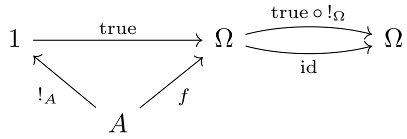
is *logical*, i.e. it preserves finite limits, exponential objects and the subobject classifier. Furthermore, if $f : A \to B$ is a morphism in $\mathcal{E}$ then the similarly defined pullback functor $f^* : \mathcal{E}/B \to \mathcal{E}/A$ is logical and has both left and right adjoints.

In the interests of space, we won't prove this. Logical functors are the natural notion of morphism between elementary topoi as they preserve all the structure in the definition; these should be contrasted with geometric morphisms discussed at the end of §1. Using this theorem, we can now go on and establish various ways in which elementary topoi distinguish themselves from other categories and look a lot like Set.

Proposition 2.2

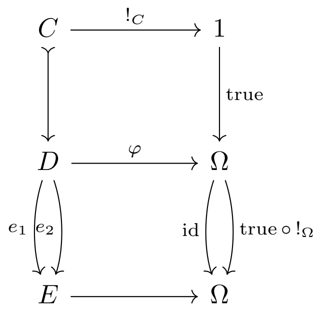
Let $\mathcal{E}$ be an elementary topos. Then we have the following statements about monos and epis:
1. All monomorphisms are equalisers.
2. $\mathcal{E}$ is a balanced category, i.e. isomorphisms are precisely the epic monos.
3. The pullback of an epimorphism is an epimorphism.
4. Any morphism $A \to 0$ into the initial object is an isomorphism.
5. Any morphism $0 \to A$ is monic.

Proof:
We first claim that the arrow $\text{true} : 1 \to \Omega$ is an equaliser. If $!_X : X \to 1$ denotes the unique arrow from an object X to the terminal object 1 then consider the following diagram:



Firstly note that $\text{true} \circ !_\Omega \circ \text{true} = \text{true} \circ !_1 = \text{id} \circ \text{true}$ so $\text{true} : 1 \to \Omega$ equalises the two arrows $\Omega \to \Omega$. Next consider the general object with a general morphism $f : A \to \Omega$ which equalises the pair, i.e. $f = \text{id} \circ f = \text{true} \circ !_\Omega \circ f = \text{true} \circ !_A$. But this precisely means that the triangle on the left commutes and $\text{true} : 1 \to \Omega$ is the equaliser of the pair of arrows on the right.

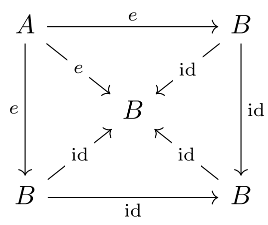
But then notice that if $C \rightarrowtail D$ is a mono then by definition of the subobject classifier we know that there is a unique $\varphi : D \to \Omega$ so that the top square in the diagram



is a pullback. The leftmost morphisms in the bottom square consist of the pullbacks of the pair of maps $\Omega \to \Omega$ also along φ. Then a quick diagram chase shows that the arrow $C \rightarrowtail D$ is the equaliser of e_1, e_2 so every mono is an equaliser in an elementary topos.

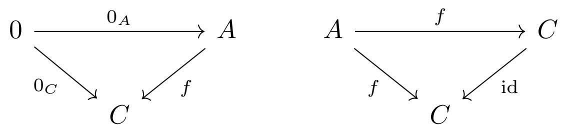
To see such a topos is balanced, notice that if an arrow $f : A \to B$ is epic and monic then it must be an epic equaliser, say of $c_1, c_2 : B \to C$. But then $c_1 \circ f = c_2 \circ f \implies c_1 = c_2$ by epicness and then since $\text{id} : B \to B$ equalises $c_1 = c_2$ it factors through f via some $g : B \to A$, i.e. $f \circ g = \text{id}$. But then $f \circ g \circ f = \text{id} \circ f = f \implies g \circ f = \text{id}$ by monicness so f is an isomorphism with inverse g.

To see that the pullback of an epimorphism $e : A \to B$ along a morphism $k : C \to B$ is epi, recall that e is epic iff the outer square in the diagram

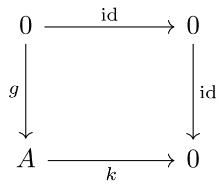


is a pushout. The morphisms inside the square then make this into a pushout square in $\mathcal{E}/B$. Taking pullbacks along the morphism $k : C \to B$ preserves small colimits, given the functor $k^* : \mathcal{E}/B \to \mathcal{E}/C$ has a right adjoint, so the pullback of e is epic in $\mathcal{E}/C$. But the forgetful functor $\mathcal{E}/C \to \mathcal{E}$ is left adjoint to the pullback functor $\mathcal{E} \to \mathcal{E}/C$ and so again preserves colimits, meaning the pullback of e along k is epic in $\mathcal{E}$.

Now let $k : A \to 0$ be a morphism into the initial object and let's think about the slice category $\mathcal{E}/0$. In a slice category $\mathcal{E}/C$, it is easy to see from thinking about the diagrams



that $0_C : 0 \to C, \mathrm{id} : C \to C$ are respectively initial and terminal objects so that in $\mathcal{E}/0$ the object $\mathrm{id} : 0 \to 0$ is both initial and terminal. Thus, since k^* preserves small limits and colimits, we know that the pullback g of $\mathrm{id} : 0 \to 0$ along $k : A \to 0$ is both initial and terminal in $\mathcal{E}/A$ and thus can be written as an arrow $g : 0 \to A$ as in the square



But then g is a pullback of an arrow which is monic and epic and so is monic and epic itself and thus an isomorphism such that $k \circ g = \mathrm{id}$. Thus k is an isomorphism as desired.

This immediately gives the last result, i.e. that any morphism $0 \to A$ is monic, since any two morphisms $B \to 0$ are isomorphisms with a common inverse $0_B : 0 \to B$ and so are the same. □

Now that we have seen that the usually rather rough analogy that epis ≈ surjections and monos ≈ injections is much stronger in elementary topoi, we might naturally wonder if the surjection-injection factorisations that exist in Set work in any such topos...

Definition 2.2. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. The **image** of f, if it exists, is the subobject $\mathrm{im}\, f \rightarrowtail B$ such that

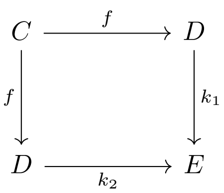
- f factors through it,
- if f factors through the subobject $S \rightarrowtail B$ then $\mathrm{im}\, f \rightarrowtail B$ also factors through S.

A standard argument tells you that this property specifies a unique subobject. Then:

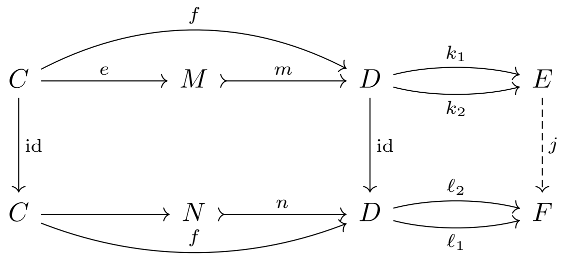
Proposition 2.3

Every morphism $f : C \to D$ in an elementary topos has a factorisation $C \to \mathrm{im}\, f \rightarrowtail D$ and furthermore the arrow $C \to \mathrm{im}\, f$ is an epimorphism.

Proof:
Monics in elementary topoi are the same thing as equalisers, as we have seen, so if we want to factor $f : C \to D$ through a monic we should look for a pair of arrows which f equalises. One good candidate is the **cokernel pair**, i.e. the morphisms $k_1, k_2 : D \to E$ which make the square



a pushout. Then construct $m : M \rightarrowtail D$ to be the equaliser of k_1, k_2 and since f equalises its cokernel pair we know that $f = m \circ e$ for some $e : C \to M$. We now claim that m is the image of f. To see this, consider another instance of f factoring through a subobject $n : N \rightarrowtail D$ as in the diagram



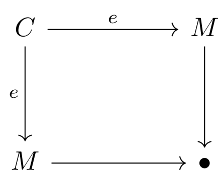
and since n is monic it is the equaliser of some arrows $\ell_1, \ell_2 : D \to F$. Since n equalises ℓ_1, ℓ_2 then so too must f and thus by definition of the cokernel pair as the pushout above there must exist a morphism $j : E \to F$ as shown such that $j \circ k_i = \ell_i$. But then

$$\ell_1 \circ m = j \circ (k_1 \circ m) = j \circ (k_2 \circ m) = \ell_2 \circ m$$

so m factors through n, the equaliser of ℓ_1, ℓ_2 thus showing m is the image.

We are left with showing e is epic. Well one thing we might try is to look at the image $m' : M' \rightarrowtail M$ of e and factorise it as $C \xrightarrow{e'} M' \xrightarrow{m'} M$. Given mm' is a monic that f factorises through, it itself must factorise through the image m of f say as $m = mm'e'' \implies 1 = m'e''$. Thus m' is epic and we already knew it was monic so it is an isomorphism.

A map whose image is an isomorphism with the codomain sounds like it must be an epimorphism! To see this and finish off, just remember that the image was the equaliser of the cokernel pair of e so the cokernel pair must be equal and the square



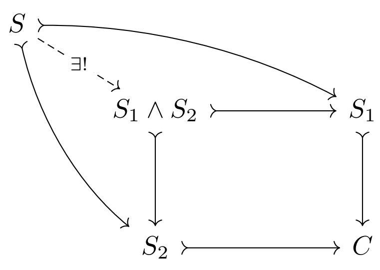
(where the unlabeled arrows are equal) is a pushout which is equivalent to e being an epimorphism. □

Finally, all of the above work allows us to state and proof the main two results of this section; this one is reminiscent of the Boolean algebra structure of the collection of subsets of a given set S.

Proposition 2.4

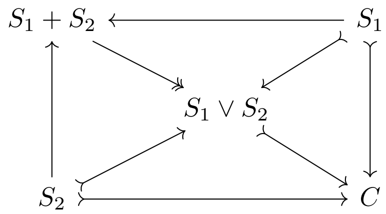
Let $\mathcal{E}$ be a well-powered topos. Then $\mathrm{Sub}_{\mathcal{E}}(C)$ for any object C is a Heyting algebra.

Proof:
To begin with, recall that $\mathrm{Sub}_{\mathcal{E}}(C)$ is a poset in any category but we need to see the existence of top and bottom elements $\top, \bot$, the lattice operations $\wedge, \vee$ and then the Heyting algebra $\implies$. The top element in the subobject poset is clearly just $C \xrightarrow{\mathrm{id}} C$ and we know from **Proposition 2.2** that the unique morphism from 0 to any object is monic so it is therefore the initial subobject of C. The operation $\wedge$ is just "take the pullback" which can be seen by considering the pullback square



We see that the morphisms $S_1 \wedge S_2 \to S_1, S_2$ are monic as they are pullbacks of monics so that $S_1 \wedge S_2$ is a subobject of C. Then note that any other subobject S of C which factors through both of S_1, S_2 factors through $S_1 \wedge S_2$ as it is their pullback over C, as desired.

$\vee$ is more complex since simply taking the coproduct of two monics into C won't necessarily produce a monic into C. The key is thus to use image factorisation!



The diagram above constructs $S_1 \vee S_2$ as the image of $S_1 + S_2 \to C$ which is clearly a subobject and the universal property of the image immediately tells us $S_1 \vee S_2$ is the least upper bound of S_1, S_2 in the poset.

To finish describing the Heyting structure, we first show that $\mathrm{Sub}_{\mathcal{E}}(1)$ has an implication operator.

If $S_1, S_2 \rightarrowtail 1$ consider that their exponential object in $\mathcal{E}$ has a unique arrow $S_1^{S_2} \to 1$ which we claim is monic. Well any arrow $A \to S_1^{S_2}$ corresponds to a unique arrow, by exponential transposition, $S_2 \times A \to S_1$ but there is only one arrow from any object X into S_1 since the composition $X \to S_1 \rightarrowtail 1$ is always the unique map $X \to 1$ and $S_1 \rightarrowtail 1$ is monic. Thus there is at most one morphism $A \to S_1^{S_2}$ making the arrow $S_1^{S_2} \to 1$ monic as desired.

That calculation also explained why $(S_2 \implies S_1) = S_1^{S_2}$ holds: $S_2 \times A = S_2 \wedge A$, since taking pullbacks over 1 is the same as taking products, and therefore the discussion showed that $A \to 1$ factors through $S_1^{S_2} \to 1$ iff $S_2 \wedge A \to 1$ factors through $S_1 \to 1$.

But for any object $C \in \mathcal{E}$ we have a natural functor $i_C : \mathrm{Sub}_{\mathcal{E}}(C) \to \mathcal{E}/C$ where subobjects $A \rightarrowtail C$ (and morphisms between them) are interpreted directly as objects (and morphisms) in $\mathcal{E}/C$, indeed subobjects of the terminal object $\mathrm{id} : C \to C$. This functor clearly by definition of $\mathcal{E}/C$ gives an isomorphism $\mathrm{Sub}_{\mathcal{E}}(C) \cong \mathrm{Sub}_{\mathcal{E}/C}(1)$ and so the former inherits from the latter an implication operator on $\mathrm{Sub}_{\mathcal{E}}(A)$ as required, making it a Heyting algebra.

□

Then it turns out that the different subobject algebras can be related using morphisms between those objects according to:

Proposition 2.5

Let $\mathcal{E}$ be a well-powered topos. For any morphism $f : C \to D$ the induced pullback functor $f^{-1} : \mathrm{Sub}_{\mathcal{E}}(D) \to \mathrm{Sub}_{\mathcal{E}}(C)$ preserves the Heyting structure and has both right and left adjoints $\forall_f, \exists_f : \mathrm{Sub}_{\mathcal{E}}(C) \to \mathrm{Sub}_{\mathcal{E}}(D)$.

Before we prove this result, the reader might be wondering where the left and right adjoints in this proposition intuitively come from and why they are labeled with the impactful symbols $\exists, \forall$. A very special case in the category Set comes from thinking about the projection $\pi_2 : X \times X \to X$ for a set X. This induces a preimage (i.e. pullback) functor on the subobject posets of the two sets which for simplicity we'll describe as a function $\pi_2^* : P(X \times X) \to PX$ between their powersets. We then claim that if we define the functions $\forall_\pi, \exists_\pi : PX \to P(X \times X)$ by

$$\forall_\pi(S) = \{x \ : \ \forall\, y, (y, x) \in S\} \qquad \exists_\pi(S) = \{x \ : \ \exists\, y,\ (y, x) \in S\}$$

then we have the adjoint relations $\exists_\pi \dashv \pi_2^* \dashv \forall_\pi$. These maps $\forall_\pi, \exists_\pi$ are fairly natural: if $R \subseteq X \times X$ is thought of as a relation and is for example a partial order on X then $\forall_\pi(R)$ is the set of top elements under this order (which is either a singleton or empty) and $\exists_\pi(R)$ is the set of elements where there's at least one smaller element. The adjunctions

$$\exists_\pi(S) \subseteq S' \iff S \subseteq \pi_2^*(S') \qquad \pi_2^*(S') \subseteq S \iff S' \subseteq \forall_\pi(S)$$

are similarly natural. $\exists_\pi$ is just the direct image and says that $\pi_2(S) \subseteq S' \iff S \subseteq \pi_2^{-1}(S')$ which is clear and the second statement says that the preimage under projection of a set S' lies in S iff S' lies in the collection of elements $x \in X$ such that $(y, x) \in S$ for all y which is again clear. These definitions and proofs can easily be extended to any function $f : A \to B$ between sets yielding $f^*, \exists_f, \forall_f$ and further to all elementary topoi.

Proof:
The inclusion functors of the form $i_C : \mathrm{Sub}_{\mathcal{E}}(C) \to \mathcal{E}/C$ actually allow us to make precise the relation between the two similar pullback functors f^{-1}, f^* for a morphism $f : A \to B$ in $\mathcal{E}$: the fact that the square

$$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{E}}(B) & \xrightarrow{f^{-1}} & \mathrm{Sub}_{\mathcal{E}}(A) \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle i_B} \\ \mathcal{E}/B & \xrightarrow{f^*} & \mathcal{E}/A \end{array}$$

commutes (by definition of f^* and the fact that $S \times 1 \cong S$). Since f^* preserves small limits, colimits and exponential objects and f^{-1} is a restriction of f^* we then get that f^{-1} indeed is a morphism of Heyting algebras.

Further, since f^* has a right adjoint and right adjoints preserve terminal objects and monomorphisms we see that its restriction to $\mathrm{Sub}_{\mathcal{E}/A}(1) \to \mathrm{Sub}_{\mathcal{E}/B}(1)$ makes sense and provides a right adjoint to f^{-1}. It is then not hard to check that the map $\exists_f : \mathrm{Sub}_{\mathcal{E}}(A) \to \mathrm{Sub}_{\mathcal{E}}(B)$ given by taking a subobject $S \rightarrowtail A$, composing it with f to get an arrow $S \to B$ and then taking its image to get a subobject of B gives a functor which is left adjoint to f^{-1}.

Alternatively, if you prefer true abstract nonsense, we can deduce the existence of left and right adjoints for f^{-1} from the above fact that it preserves all small limits and colimits and then applying the Adjoint Functor Theorem for preorders.

3 Elementary vs. Grothendieck

In this section, we discuss some basic examples of elementary topoi and in particular prove that all Grothendieck topoi are elementary. We then contrast Grothendieck with elementary topoi with the existence of all small colimits and the distinction between the natural morphisms between them.

Let's first look at examples of elementary topoi.

Example 3.1. The categories Set and FinSet are both elementary topoi. Indeed, they both have all finite limits, exponentials and they share $\{0, 1\}$ and the function true : $\{*\} \to \{0, 1\}$ sending $* \mapsto 1$ as subobject classifier. The example of FinSet shows, however, that there's no requirement for arbitrary small limits or colimits to exist in an elementary topos.

Example 3.2. The category of G-sets, i.e. sets equipped with a right action of the group G with equivariant maps between them or equivalently functors $\mathrm{B}G^{\mathrm{op}} \to \mathrm{Set}$ with natural transformations between them, can be checked to be an elementary topos. Clearly this category has all finite limits given that in functor categories limits are taken pointwise and Set has all limits.

Next, let's consider what subobjects are in this category. Given limits are taken pointwise in this category, monomorphisms $S' \rightarrowtail S$ are precisely those whose underlying equivariant function is injective. Thus subobjects of S can be identified with the subsets $S' \subseteq S$ which are closed under the action of G, i.e. $s \in S' \implies s \cdot g \in S'$ for all $g \in G$.

This makes the subobject classifier very simple: just take the arrow true : $\{*\} \to \{0, 1\}$ which sends $* \mapsto 1$ as before and give both of these sets the trivial right G-action (i.e. all elements of G fix all elements of these sets). This is because equivariant characteristic maps $S \to \{0, 1\}$ correspond bijectively with subsets of S closed under the action of G.

To show that this category has all exponential objects Y^X for G-sets X, Y recall that any equivariant map $f : X \times Z \to Y$ (for Z some G-set) gives *some function* $Z \to Y^X$ by sending $z \mapsto f_z(-) := f(-, z)$. Thus a natural guess is to make the exponential object Y^X in this category have the usual underlying set, i.e. the set of all functions $X \to Y$, but we need to choose a right action on this set so that the map $z \mapsto f_z(-)$ is equivariant. Well

$$f_{z\cdot g}(x) = f(x, z \cdot g) = f((x \cdot g^{-1}) \cdot g, z \cdot g) = f(x \cdot g^{-1}, z) \cdot g = f_z(x \cdot g^{-1}) \cdot g$$

using the equivariance of f. Thus if we make the action on elements $f \in Y^X$ by an element $g \in G$ be $(f \cdot g)(-) := f((-) \cdot g^{-1}) \cdot g$ then the map $z \mapsto f_z$ becomes equivariant and indeed Y^X is an exponential object in the category of G-sets as required.

Notice that all of the above work can be done similarly in the category of finite G-sets, so that too is an elementary topos.

We now want to show that all Grothendieck topoi are elementary inspired by the proofs in [12] (Chapter III) and [10] (Chapter 2), subsuming the last example and giving a great wealth of others. For this, we need a quick detour into some quick background general category theory.

Definition 3.1. Let $R : \mathcal{C}^{\text{op}} \to \text{Set}$ be a presheaf. Its associated **category of elements** $\int R$ has as objects pairs (C, x) with C an object of $\mathcal{C}$ and $x \in R(C)$ and as morphisms $f : (C, x) \to (D, y)$ morphisms $f : C \to D$ in $\mathcal{C}$ such that $R(f)(y) = x$.

This category naturally comes equipped with a forgetful functor $\pi_R : \int R \to \mathcal{C}$ sending $(C, x) \mapsto C$ and $(f : (C, x) \to (D, y)) \mapsto (f : C \to D)$. Then:

Lemma 3.1

Every presheaf $R : \mathcal{C}^{\text{op}} \to \text{Set}$ is a colimit of representable functors. In particular,

$$R \cong \varinjlim_{(C,x)} (y \circ \pi_R)$$

Proof:
We prove this using the Yoneda Lemma, i.e. we hope to show that in $[\mathcal{C}^{\text{op}}, \text{Set}]$ there is a natural isomorphism $\text{Hom}\left(\varinjlim_{(C,x)}(y \circ \pi_R), -\right) \cong \text{Hom}(R, -)$. Well note that for each presheaf $P : \mathcal{C}^{\text{op}} \to \text{Set}$ we have

$$\text{Hom}\left(\varinjlim_{(C,x)} (y \circ \pi_R), P\right) \cong \varprojlim_{(C,x)} \text{Hom}(y(C), P) \cong \varprojlim_{(C,x)} P(C)$$

where the last isomorphism uses Yoneda and the change from $\varinjlim$ to $\varprojlim$ reflects the fact that $\text{Hom}(-, P) : \mathcal{C}^{\text{op}} \to \text{Set}$ preserves colimits in the sense that colimit diagrams in $\mathcal{C}$, which are limit diagrams in $\mathcal{C}^{\text{op}}$, are sent to limits in Set. But then we can write this limit explicitly as

$$\varprojlim_{(C,x)} P(C) \cong \text{Hom}_{[(\int R)^{\text{op}}, \text{Set}]}(\text{pt}, P)$$

where $\text{pt} : \left(\int R\right)^{\text{op}} \to \text{Set}$ is the presheaf sending every object to a singleton. To see that the RHS of this isomorphism has the universal property of the LHS, notice simply that for any cone $(\lambda_{(C,x)} : S \to P(C))$ over the diagram $P \circ \pi_R$ the collection of images of an element $s \in S$ induces a natural transformation $\lambda^s : \text{pt} \implies P$ with component maps the restrictions $\lambda^s_{(C,x)} : \{s\} \to P(C)$ and this defines the necessary function $S \to \text{Hom}_{[(\int R)^{\text{op}}, \text{Set}]}(\text{pt}, P)$ sending $s \mapsto \lambda^s$ demonstrating the universal property.

Finally we claim that there's a natural isomorphism

$$\text{Hom}_{[(\int R)^{\text{op}}, \text{Set}]}(\text{pt}, P) \cong \text{Hom}_{[\mathcal{C}^{\text{op}}, \text{Set}]}(R, P)$$

In particular, given a natural transformation $\alpha : \mathrm{pt} \implies P$ we get an induced function $f^\alpha_C : R(C) \to P(C)$ for each object C of $\mathcal{C}$ given by $f^\alpha_C(x) := \alpha_{(C,x)}(\{\mathrm{pt}\})$. It is simple to check that the naturality of α implies $(f^\alpha_C : R(C) \to P(C))$ together define a natural transformation $f^\alpha : R \implies P$ and that the assignment $\alpha \mapsto f^\alpha$ gives an isomorphism between the two above sets. □

This allows us to finally prove:

Theorem 3.1

All Grothendieck topoi are elementary.

Proof:
We need to check they have:

- all finite limits,
- all exponential objects.
- a subobject classifier,

In each case, we will first show that it's true for a presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ and then a general Grothendieck topos $\mathrm{Sh}(\mathcal{C}, \mathcal{J})$. The first point is clear as we can recall that limits and colimits are pointwise in functor categories so that since Set has all small limits and colimits so too does $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. To show this for $\mathrm{Sh}(\mathcal{C}, \mathcal{J})$, we now claim that a limit of sheaves taken in the presheaf category is also a sheaf.

Let $P = \varprojlim P_i$ be a limit of sheaves P_i; since limits are pointwise we know that for each object C we have $P(C) = \varprojlim P_i(C)$. Now we know from §1 that being a sheaf can be expressed as an equaliser condition for each covering sieve $\mathcal{S}$ and we know it is satisfied by each P_i, i.e.

$$P_i(C) \xrightarrow{\ e\ } \prod_{(f:D\to C)\in\mathcal{S}} P_i(D) \underset{p}{\overset{m}{\rightrightarrows}} \prod_{(f:D\to C)\in\mathcal{S},g:E\to D} P_i(E)$$

is an equaliser diagram for each P_i (for the appropriate maps m, p defined above). But limits commute with limits, so the limit of all the above kind of equalisers yields an equaliser

$$P(C) \xrightarrow{\ e\ } \prod_{(f:D\to C)\in\mathcal{S}} P(D) \underset{p}{\overset{m}{\rightrightarrows}} \prod_{(f:D\to C)\in\mathcal{S},g:E\to D} P(E)$$

Given that this can be done for all covering sieves $\mathcal{S}$ we see that P is a sheaf and so $\mathrm{Sh}(\mathcal{C}, \mathcal{J})$ has all small limits.

To show that $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ has all exponential objects, our first guess might be that the exponential P^Q of two presheaves $P, Q : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is pointwise but some thought makes us realise that there's no obvious way to make $P^Q(C) = \mathrm{Hom}(QC, PC)$ into a functor. To remedy this, we can reason that if $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is the Yoneda embedding then if P^Q exists we would definitely have natural isomorphisms

$$P^Q(C) = \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(y(C), P^Q) \cong \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(Q \times y(C), P)$$

where the first isomorphism is due to the Yoneda lemma. It is then not hard to check that this works as a *definition* of an exponential object by using the above fact that every presheaf

P is a colimit $\varinjlim y(C_i)$ of representable functors and then calculating

$$\begin{aligned}\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(R, P^Q) &\cong \mathrm{Hom}(\varinjlim y(C_i), P^Q)\\ &\cong \varinjlim \mathrm{Hom}(y(C_i), P^Q)\\ &\cong \varinjlim \mathrm{Hom}(Q \times y(C_i), P)\\ &\cong \varinjlim \mathrm{Hom}(Q \times \varinjlim y(C_i), P)\\ &\cong \mathrm{Hom}(Q \times R, P)\end{aligned}$$

where the isomorphism (2) $\cong$ (3) is by definition of P^Q and (3) $\cong$ (4) comes from the fact that $Q \times (-)$ preserves colimits as colimits & products are taken pointwise in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ and products preserve colimits in Set. Thus P^Q is indeed an exponential object in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. $\mathrm{Sh}(\mathcal{C}, \mathcal{J})$ then has all exponential objects as it can be checked that if P is a sheaf then P^Q is always a sheaf under this construction.

To show that $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ has a subobject classifier, we again use representables see that if Ω exists we must have

$$\Omega(C) \cong \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(y(C), \Omega) \cong \{\text{subpresheaves of } y(C)\}$$

where the first isomorphism is due to the Yoneda embedding. It is customary to then consider the fact that the RHS has a natural bijection with sieves on C given by

$$(P \subseteq y(C)) \mapsto S := \{f \ : \ f \in P(A) \text{ where } f : A \to C\}$$

$$\mathcal{S} \text{ sieve on } C \mapsto P : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}, P(D) := \{f \ : \ f : A \to C, f \in \mathcal{S}\}$$

and then you define $\Omega(C) := \{\text{sieves on } C\}$. It is then easy to check that this is a subobject classifier for all presheaves in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ using the colimit-of-representables idea as above and doing some simple calculations. For $\mathrm{Sh}(\mathcal{C}, \mathcal{J})$, it can be checked that $\Omega(C) := \{\text{subpresheaves of } y(C) \text{ which are sheaves}\}$ is a sheaf and works as a subobject classifier. □

The above proof demonstrates within it a key difference between elementary and Grothendieck topoi: Grothendieck topoi have all small limits and thus, thanks to **Theorem 2.1**, all small colimits. This immediately tells us that the examples above of FinSet and G-sets cannot be Grothendieck.

4 Interpreting Logic

In this section we briefly discuss the vision of Lawvere and Tierney: how to interpret logics in elementary topoi. We first describe our languages of interest and then describe how to define interpretations of languages in elementary topoi, generalising $\mathcal{L}$-structures, as well as models of theories. We will see that interpretations are functorial in a natural sense and logical functors preserve all models while geometric morphisms only preserve models in a certain fragment of logic, geometric logic.

Definition 4.1 (Language)**.** A language $\mathcal{L}$ for us comes with:

- A collection of basic *types* or *sorts* $X_1, X_2, \dots$ and for each sort Y, infinitely many variables $v_1^Y, v_2^Y, \dots$ (we allow variables $(z_1, \dots, z_n)$ for types $Z_1 \times \dots \times Z_n$). If Z_1, Z_2 are sorts then so are the expressions $Z_1 \times Z_2$ and PZ_1.
- Relation symbols $R, S \dots$ which also come with the data of the types of their inputs, written $R \subseteq Z_1 \times \dots Z_n$ if the suggested inputs will be of types $Z_1, \dots, Z_n$.
- Similarly we have function symbols $f, g, \dots$ along with their types written $f : Z_1 \times \dots \times Z_n \to Y$ for an n-ary function with input types $Z_1, \dots, Z_n$ and output type Y.

Given a language $\mathcal{L}$, we build up terms and formulas in the traditional inductive manner:

- Every variable and constant (i.e. function symbol of arity zero) is a term of their assigned types. If $t_1, \dots, t_n$ are terms of types $Z_1, \dots, Z_n$ and $f : Z_1 \times \dots \times Z_n \to Y$ is a function symbol then $f(t_1, \dots, t_n)$ is a term of type Y.
- $\top, \bot$, $R(t_1, \dots, t_n)$ for terms $t_1, \dots t_n$ of types $Z_1, \dots, Z_n$ and a relation symbol $R \subseteq Z_1 \times \dots \times Z_n$ and $s_1 = s_2$ for two terms s_1, s_2 of a common type together make up the atomic formulae. General formulae are then built up from these in the normal manner using $\wedge, \vee, \neg, \implies$ and quantifier expressions $(\exists\, x \in X), (\forall\, x \in X)$ for any sort X.
- One difference to point out is that we allow the formation of infinite conjunctions $\bigvee_{i \in I} \varphi_i$ and disjunctions $\bigwedge_{i \in I} \varphi_i$ for I a set and φ_i each formulae already constructed.

The first important thing to note is the inclusion of PX for each sort X; this will be interpreted by the powerset functor defined above and it allows us to express higher order logic as quantifying over PX corresponds to being allowed to quantify over subsets of X (at least in Set). Another thing is that a lot of this definition is redundant as we will in the sequel interpret formulae $\varphi(\mathbf{x})$ as objects suggestively written as $\{\mathbf{x} : \varphi(\mathbf{x})\}$ and using this we can write definitions of expressions which look like the set-theoretic definition of, say, B^A for sorts B, A and then define function symbols as constants of type B^A. We don't do this here out of ease[1], similarity to traditional theory and because for the most part we are only interested in first-order theories. Now for semantics!

Definition 4.2. Let $\mathcal{L}$ be a language and $\mathcal{E}$ be an elementary topos. An **interpretation** M of $\mathcal{L}$ in $\mathcal{E}$ consists of:

- an assignment of an object $X^{(M)}$ of $\mathcal{E}$ for each basic sort X of $\mathcal{L}$ with $(Z_1 \times \dots \times Z_n)^{(M)} := Z_1^{(M)} \times \dots \times Z_n^{(M)}$ and $(PX)^{(M)} := P(X^{(M)})$ for $P : \mathcal{E}^{\text{op}} \to \mathcal{E}$ the powerset functor,
- a subobject $R^{(M)} \rightarrowtail Z_1^{(M)} \times \dots \times Z_n^{(M)}$ for each relation symbol $R \subseteq Z_1 \times \dots \times Z_n$,
- an arrow $f^{(M)} : Z_1^{(M)} \times \dots \times Z_n^{(M)} \to Y^{(M)}$ for a function symbol $f : Z_1 \times \dots \times Z_n \to Y$

A **theory** in the language $\mathcal{L}$ is a collection of formulas from that language.

An interpretation as defined gives semantic meaning in $\mathcal{E}$ of all the basic structures in the language but of course this induces semantics on much more. In the sequel, $\mathbf{z} := (z_1, \dots, z_n)$ which is a collection of variables of types $Z_1, \dots, Z_n$ and $\varphi(\mathbf{z}) := \varphi(z_1, \dots, z_n)$ for a formula/term φ where we assume that $(z_1, \dots, z_n)$ contains all the free variables in φ. Then we can assign to each term $t = t(\mathbf{z})$ an arrow:

- if $t = x_i$ is a variable then $t^{(M)}(\mathbf{z})$ is the i-th projection $\pi_i : Z_1^{(M)} \times \dots \times Z_n^{(M)} \to Z_i^{(M)}$
- if $t = f(t_1, \dots, t_m)$ for $f : Y_1 \times \dots \times Y_k \to Y$ then $t^{(M)}(\mathbf{z})$ is the obvious composite $Z_1^{(M)} \times \dots \times Z_n^{(M)} \xrightarrow{\langle t_1, \dots, t_m \rangle} Y_1^{(M)} \times \dots \times Y_k^{(M)} \xrightarrow{f^{(M)}} Y$.

Much more interesting, we can interpret each formula $\varphi(\mathbf{z})$ as a subobject $\{\mathbf{z} : \varphi\}^{(M)} \rightarrowtail Z_1^{(M)} \times \dots \times Z_n^{(M)}$, starting with atomic formulae:

[1] However the PhD thesis [1], for example, does this.

- The symbols $\top, \bot$ are interpreted as the top and bottom subobjects always, i.e. $Z_1^{(M)} \times \ldots \times Z_n^{(M)}$ and 0 respectively.
- The formula $t = t'$ for terms t, t' is interpreted by the equaliser of the arrows $t^{(M)}, t'^{(M)}$.
- The formula $R(t_1, \ldots, t_k)$ for $R \subseteq Y_1 \times \ldots \times Y_k$ and t_i having sort Y_i has interpretation $\{\mathbf{z} : R(t_1, \ldots, t_k)\}^{(M)}$ being the pullback

$$\begin{array}{ccc} \{\mathbf{z} : R(t_1, \ldots, t_k)\}^{(M)} & \longrightarrow & R^{(M)} \\ \downarrow & & \downarrow \\ X_1^{(M)} \times \ldots \times X_n^{(M)} & \xrightarrow{\langle t_1^{(M)}, \ldots, t_k^{(M)} \rangle} & Y_1^{(M)} \times \ldots \times Y_k^{(M)} \end{array}$$

We can then inductively define subobjects for formulae built out of the connectives $\vee, \wedge, \implies, \neg$ using the Heyting algebra operations we constructed in **Proposition 2.5** (where $\neg(-) := (-) \implies \bot$ as per usual) and finally we define the interpretations of quantifiers on subobjects of $Z_1^{(M)} \times \ldots \times Z_n^{(M)}$ as

$$\{\mathbf{z} \ : \ (\forall\, x \in X)\, \varphi(\mathbf{z}, x)\}^{(M)} = \forall_\pi \left(\{(\mathbf{z}, x) \ : \ \varphi(\mathbf{z}, x)\}^{(M)} \right)$$

$$\{\mathbf{z} \ : \ (\exists\, x \in X)\, \varphi(\mathbf{z}, x)\}^{(M)} = \exists_\pi \left(\{(\mathbf{z}, x) \ : \ \varphi(\mathbf{z}, x)\}^{(M)} \right)$$

where $\forall_\pi, \exists_\pi : \mathrm{Sub}_{\mathcal{E}}(Z_1^{(M)} \times \ldots \times Z_n^{(M)} \times X^{(M)}) \to \mathrm{Sub}_{\mathcal{E}}(Z_1^{(M)} \times \ldots \times Z_n^{(M)})$ are the right & left adjoints (also constructed in **Proposition 2.5**) to the pullback functor induced by the projection $\pi : Z_1^{(M)} \times \ldots \times Z_n^{(M)} \times X^{(M)} \to Z_1^{(M)} \times \ldots \times Z_n^{(M)}$. This explains all the work we did in the previous section: we can now give meaning to quantifiers in a general elementary topos!

Definition 4.3. A formula φ is said to be **valid** or **satisfied** in an interpretation M of a language $\mathcal{L}$ if every subobject $\{\mathbf{z} : \varphi\}^{(M)} \rightarrowtail Z_1^{(M)} \times \ldots \times Z_n^{(M)}$ is the identity map or the top subobject. M is then said to be a **model** of a theory T if all the formulae in T are valid in M.

For a logician, this should pretty mind-expanding. The languages we have described are highly expressive, allowing for quantification to arbitrary order-depth (e.g. quantifying over sets of subsets of a given domain etc.) and allowing reference to arbitarily many types, and we have set up a brave new world of semantics where models can be considered in all kinds of topoi beyond simply Set. We can think of other elementary topoi as being like other set theoretic universes, in this sense, which in general are not governed by classical but intuitionistic logic.

Let's look at some examples of this setup.

Example 4.1 (Local Rings)**.** In the interest of the geometric theme, let's consider axiomatising local rings. Our language will be the familiar language of rings: one sort R, three function symbols $+, \cdot : R \times R \to R, - : R \to R$ and two constants $0, 1 \in R$ (i.e. function symbols $1 \to R$). We then have the usual axioms of rings saying that $(\forall\, r \in R)\, r + 0 = r$, commutativity of addition and so forth and then additionally we add the axiom

$$(\forall\, r \in R)\, ((\exists\, s \in R) r \cdot s = 1 \vee (\exists\, s \in R)(1 - r) \cdot s = 1)$$

This condition, that one of $r, 1 - r$ is a unit for all $r \in R$, is equivalent to there being a unique maximal ideal of R but is simpler to work with, given it is first order.

Let's unpack what a model M in a topos $\mathcal{E}$ of this theory is in more detail. We get an object $R^{(M)}$ for our type R, arrows for our function symbols etc. so let's see what it means for the formula $(\forall\, a \in R)(\forall\, b \in R)\; a + b = b + a$ to be valid. It is a formula with no free variables and so its validity is equivalent to it being the top subobject of 1. So we can compute (suppressing the (M)-notation)

$$\{\ :\ (\forall\, a \in R)(\forall\, b \in R)\; a + b = b + a\} = 1 \iff \forall_{\pi_1} (\forall_{\pi_0} (\{(a, b) \ : \ a + b = b + a\})) = 1$$

$$\iff \{(a,b) \ : \ a+b=b+a\} = R \times R$$

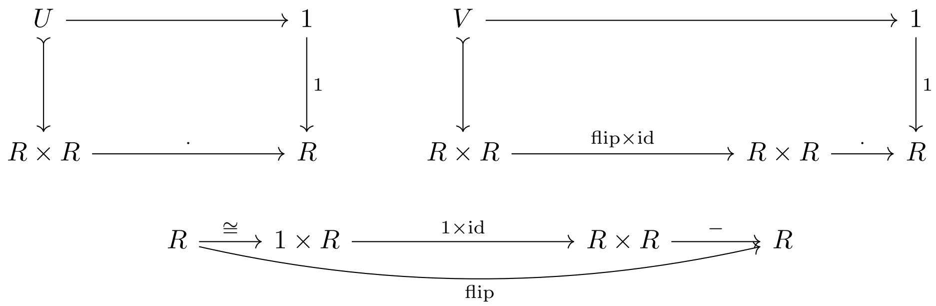
where the first equivalence is by the definition of interpretations and then the second equivalence comes from the fact that both $\pi^*, \forall_\pi$ for the projections π preserve limits (given they are both right adjoints). The final subobject is then the equaliser of the maps $R \times R \xrightarrow{+} R$ and $R \times R \xrightarrow{\text{twist}} R \times R \xrightarrow{+} R$ (where twist exchanges the two factors), which we require to be all of $R \times R$. Similarly the locality axiom above is equivalent to

$$\exists_{\pi_1}(\{(r,s) \ : \ r \cdot s = 1\}) \vee \exists_{\pi_1}(\{(r,s) \ : \ (1-r) \cdot s = 1\}) = R$$

Since $\exists_{\pi_1} : \mathrm{Sub}_{\mathcal{E}}(R \times R) \to \mathrm{Sub}_{\mathcal{E}}(R)$ is a left adjoint, we can bring the lattice-union $\vee$ inside and given it takes the image of the post-composition of a subobject with $\pi_1 : R \times R \to R$, the above is equivalent to the statement that the two subobjects of $R \times R$

$$U = \{(r,s) \ : \ r \cdot s = 1\} \rightarrowtail R \times R \qquad V = \{(r,s) \ : \ (1-r) \cdot s = 1\} \rightarrowtail R \times R$$

have the property that the coproduct of the arrows $U, V \rightarrowtail R \times R \xrightarrow{\pi_1} R$ is an epimorphism (so that the image is all of R). Finally, these U, V are defined by pullbacks/equalisers



with flip : $R \to R$ defined as shown, encoding the map $r \mapsto (1-r)$ using the given maps.

Example 4.2 (Internal Language)**.** Another important example of the above ideas is the *internal language* of an elementary topos: if topoi behave a lot like the category Set, why not make a language which allows us to directly describe with formulae subobjects in the elementary topos we want to work with? There is a type X for every object X in the category and a function symbol $f : C \to D$ for every arrow $f : C \to D$; in general terms σ of type X will be arrows $\sigma : U \to X$, where U is the product of the types of the free variables of σ, and we will think of the syntax and actual arrows as interchangeable.

Some of the notation used for this language is idiosyncratic which we'll now describe. For instance, formulas in this language are going to be arrows in the category like anything else and so in particular they are the terms of type Ω. We now describe certain term formation rules and their interpretations simultaneously:

- If $\sigma : U \to Z_1, \tau : V \to Z_2$ are terms of types Z_1, Z_2 then we get a term $\langle \sigma, \tau, \rangle$ of type $Z_1 \times Z_2$ whose interpretation an arrow is $\langle \sigma \circ p_1, \tau \circ p_2 \rangle : W \to X \times Y$ where $p_1 : W \to U, p_2 : W \to V$ are projections; we write it this way because $\langle \sigma, \tau \rangle$ has as its free variables the union of those of σ, τ so the domain W might not have $W \cong U \times V$. For example if $\sigma : X_1 \times X_2 \to Z_1, \tau : X_2 \times X_3 \to Z_2$ had free variables x_1, x_2 and x_2, x_3 respectively then $W = X_1 \times X_2 \times X_3$ and we'll continue to use this notation from now on.
- If $\sigma : U \to X, \tau : V \to X$ are terms of the same type, we can get a term $\sigma = \tau$ of type Ω which is given by the composite
$$W \xrightarrow{\langle \sigma p_1, \tau p_2 \rangle} X \times X \xrightarrow{\chi_{\Delta_X}} \Omega$$
where $\chi_{\Delta_X} : X \times X \to \Omega$ is the characteristic morphism associated to the diagonal subobject $\Delta_X : X \rightarrowtail X \times X$.
- We then need an internal description of function *application*: if $f : V \to Y^X, \sigma : U \to X$ are

terms of the shown types then we can form an application term $f(\sigma)$ of type Y given as the composite

$$W \xrightarrow{\langle f \circ p_1, \sigma \circ p_2 \rangle} Y^X \times X \xrightarrow{\text{eval}} Y$$

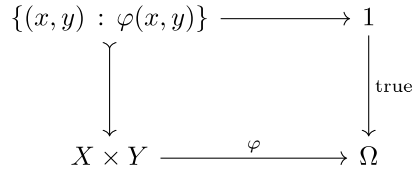
In the special case where $Y = \Omega$, this is written sometimes as $\sigma \in f$. Also note that arrows $f : X \to Y$ can be equivalently thought of as their transposes $\hat{f} : 1 \to Y^X$ and it is then easy to check that $\hat{f}(\sigma) = f \circ \sigma$ so often this is also notated by $f(\sigma)$.

- Finally we have the internal exponential transposition on terms where if $\sigma : X \times U \to Z$ is a term of type Z and x is a variable of type X (interpreted as defined above by an identity arrow $X \to X$) then we can form a term $\lambda x \sigma$ of type Z^X given as an arrow $U \to Z^X$ by the transpose of σ.

To get the connectives $\wedge, \vee$ etc. on formulas, consider the fact that the Heyting structure on each subobject poset in $\mathcal{E}$ is defined in a uniform manner which is irrespective of the choice of object and in particular reexamination of the proof of **Proposition 2.4** shows that each of these give natural transformations $\mathrm{Sub}_{\mathcal{E}}(-) \times \mathrm{Sub}_{\mathcal{E}}(-) \to \mathrm{Sub}_{\mathcal{E}}(-)$ and thus morphisms $\Omega \times \Omega \to \Omega$ (or just $\Omega \to \Omega$ in the case of $\neg$) recalling that it is the representing object of the subobject functor and applying the Yoneda lemma. For example, if φ, ψ are formulas or terms of type Ω we define $\varphi \wedge \psi$ as the composite

$$W \xrightarrow{\varphi \circ p_1, \psi \circ p_2} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

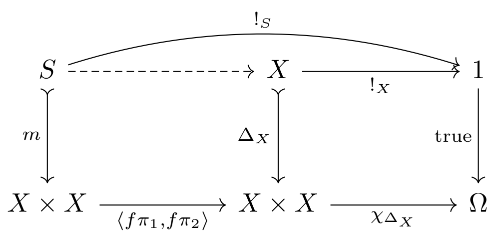
We can then recover our way of considering formulas as subobjects like above via the defining property of the subobject classifier, e.g. if $\phi(x, y)$ is a formula with free variables $X \times Y$ then $\{(x, y) \ : \ \varphi(x, y)\}$ is the subobject



and then applying the functors $\forall_{\pi_1}, \exists_{\pi_1}$ as above induced by the projection $\pi_1 : X \times Y \to X$ allows us to get the subobjects of X associated to $(\forall x \in X)\varphi(x, y), (\exists x \in X)\varphi(x, y)$ and these formulas are those subobjects' characteristic morphisms $\to \Omega$.

The internal language allows us to describe and prove facts about the topos $\mathcal{E}$ in a much simpler language which resembles working with sets. We shall return to this idea in §7 with a more complex case study but to illustrate we consider the fact that a morphism $f : X \to Y$ in $\mathcal{E}$ is a monomorphism iff the formula $(\forall x_1 \in X)(\forall x_2 \in X)\ (f(x_1) = f(x_2) \implies x_1 = x_2)$ is valid!

To see this, note that as usual the formula is valid iff $\{(x_1, x_2) \ : \ f(x_1) = f(x_2) \implies x_1 = x_2\} \rightarrowtail X \times X$ is the top subobject and this holds iff we have the inequality of subobjects $\{(x_1, x_2) \ : \ f(x_1) = f(x_2)\} \leq \{(x_1, x_2) \ : \ x_1 = x_2\}$. Well we first claim that the former subobject (which we'll call S) is, similar to the interpretations above, the equaliser of the maps $f\pi_1, f\pi_2 : X \times X \to X$ since looking at the diagram



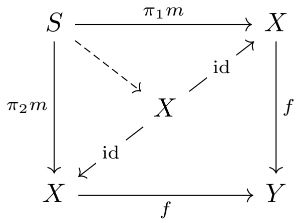
we know that the outermost rectangle and rightmost squares are pullbacks by definition and we get the map $S \to X$ by the universal property of the diagonal being a pullback. But the composite $S \to X \xrightarrow{\Delta_X} X \times X$ has its projections to each X factor being equal so similarly we get that

$$\pi_1 \left(\langle f\pi_1, f\pi_2 \rangle \circ m \right) = \pi_2 \left(\langle f\pi_1, f\pi_2 \rangle \, m \right) \iff f\pi_1 m = f\pi_2 m$$

Thus $S \xrightarrow{m} X \times X$ equalises $f\pi_1, f\pi_2$ and a very similar argument using the fact that the outermost rectangle is a pullback shows that it is these maps' equaliser which is equivalent to

$$\begin{array}{ccc} S & \xrightarrow{\pi_1 m} & X \\ {\scriptstyle \pi_2 m}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

being a pullback square. Well notice then that the subobject $\{(x_1, x_2) \, : \, x_1 = x_2\} \rightarrowtail X \times X$ is the diagonal $\Delta_X : X \rightarrowtail X \times X$ so the inequality $S \leq \Delta_X$ is equivalent to there existing a dashed map in



making the diagram commute which is itself equivalent to the statement that $\pi_1 m = \pi_2 m$. But a square of the form

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback iff g is monic so at last we see that

$$(\forall x_1 \in X)(\forall x_2 \in X)\ (f(x_1) = f(x_2) \implies x_1 = x_2) \text{ valid } \iff f \text{ monic}$$

The above examples reduce rather complex categorical statements/conditions into syntactic formulae which look much more familiar and they beg the question: if you were to work directly with the formulae in your proof system of choice and deduce, say, that $(\forall\, r \in R)\, r \cdot 0 = 0$ for R a ring object would it mean that this formula is valid in a model of your theory? The answer is yes *as long as you reason intuitionistically*; we shall not discuss it in this essay but utile proof systems for the sorts of intuitionistic logic used in topoi can be developed and soundness and completeness theorems proven (e.g. a theorem holds in propositional intuitionistic logic iff it holds in all elementary topoi's internal logics).

Now that we have defined and discussed examples of models, it's natural to consider morphisms between them. From now on note that we will only be discussing first-order theories as they are the subject of the theory of classifying topoi and still allow us to discuss many objects of interest (e.g. the local rings above); notably this means that languages, interpretations etc. no longer need to make reference to power objects.

Definition 4.4. A **homomorphism** between two interpretations M, M' of a language $\mathcal{L}$ in an elementary topos $\mathcal{E}$ is a collection of morphisms $H_X : X^{(M)} \to X^{(M')}$ for each type X such that the bottom and rightmost diagrams in

$$\begin{array}{ccc} R^{(M)} & \rightarrowtail & Z_1^{(M)} \times \ldots \times Z_n^{(M)} \\ \downarrow & & \downarrow{\scriptstyle H_{Z_1} \times \ldots \times H_{Z_n}} \\ R^{(M')} & \rightarrowtail & Z_1^{(M')} \times \ldots \times Z_n^{(M')} \end{array} \qquad \begin{array}{ccc} Z_1^{(M)} \times \ldots \times Z_n^{(M)} & \xrightarrow{f^{(M)}} & Y^{(M)} \\ \downarrow{\scriptstyle H_{Z_1} \times \ldots \times H_{Z_n}} & & \downarrow{\scriptstyle H_Y} \\ Z_1^{(M')} \times \ldots \times Z_n^{(M')} & \xrightarrow{f^{(M')}} & Y^{(M')} \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{c^{(M)}} & X^{(M)} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle H_X} \\ 1 & \xrightarrow{c^{(M')}} & X^{(M')} \end{array}$$

commute; the bottom displays the special case of the rightmost for zero-arity function symbols $1 \xrightarrow{c} X$ explicitly. The leftmost diagram expresses the fact that the composite arrow $R^{(M)} \to Z_1^{(M')} \times \ldots \times Z_n^{(M')}$ displayed must factor through the subobject $R^{(M')}$.

This makes the collection of interpretations of a language and theory in a category $\mathcal{E}$ into a category itself and we notate the full subcategory of models of a theory T by $\mathrm{Mod}(T, \mathcal{E})$. A natural question then is whether or not, if we fix a theory T, $\mathrm{Mod}(T, -)$ can be made into a functor!

Well the first observation is that functors $F : \mathcal{E} \to \mathcal{F}$ between topoi which preserve finite limits, often called *left exact* due to inspiration from homological algebra, carry an $\mathcal{E}$-interpretation M of a language $\mathcal{L}$ to an $\mathcal{F}$-interpretation $F(M)$ in a natural way:

- It assigns objects as $F(X)^{F(M)} := F(X^{(M)})$,
- relations as $R^{F(M)} := F(R^{(M)}) \rightarrowtail F(Z_1^{(M)} \times \ldots \times Z_n^{(M)}) \cong Z_1^{F(M)} \times \ldots \times Z_n^{F(M)}$ (where the monomorphism and product are preserved since F is left exact),
- arrows as $f^{F(M)}$ being the composite

$$Z_1^{F(M)} \times \ldots \times Z_n^{F(M)} \cong F(Z_1^{(M)} \times \ldots \times Z_n^{(M)}) \xrightarrow{Ff^{(M)}} F(Y^{(M)}) =: Y^{F(M)}$$

It is then not hard to see by induction on the construction of a term $t(\mathbf{z})$ that under this interpretation $F(M)$ the square

$$\begin{array}{ccc} F(Z_1^M \times \ldots \times Z_n^{(M)}) & \xrightarrow{Ft^{(M)}} & F(Y^{(M)}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} \\ Z_1^{F(M)} \times \ldots \times Z_n^{F(M)} & \xrightarrow{t^{F(M)}} & Y^{F(M)} \end{array}$$

always commutes. This observation applies to the inverse image $f^* : \mathcal{E} \to \mathcal{F}$ of any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ as left exactness was part of their definition. The issue is that although it might carry interpretations to interpretations it might fail to carry *models to models*. However, it is easy to verify that this does pleasantly happen for theories written in a large fragment of first order logic:

Definition 4.5 (Geometric Logic)**.** Let $\mathcal{L}$ be a language. A formula φ is said to be **geometric** iff it can be formed from atomic formulas via finitary use of $\wedge$, potentially infinitary use of $\vee$ and existential quantifications $(\exists\, x \in X)$. A **geometric theory** is one where all the axioms are all the form

$$(\forall\, z_1 \in Z_1)(\forall\, z_2 \in Z_2)\ldots(\forall\, z_n \in Z_n)(\varphi(\mathbf{z}) \implies \psi(\mathbf{z}))$$

with φ, ψ geometric formulae.

Many theories that you are familiar with (rings, groups, partial orders etc.) can be defined using geometric theories and, if you try, you might find that while your initial attempts at axiomatising your favourite object are not geometric it's always possible to rephrase them in a classically equivalent way which is of the above form. This is due to the fact that there is a process which can convert an finitary first order theory (i.e. one which only has finitary applications of $\wedge, \vee$) into a finitary geometric theory (called a **coherent theory**) which has the same models in Set called its **Morleyisation** ([9], Section D3.5.1), but we won't discuss this further here.

Proposition 4.1

Let $\mathcal{L}$ be a language, $f : \mathcal{F} \to \mathcal{E}$ a geometric morphism between two Grothendieck topoi and M an interpretation of $\mathcal{L}$. Then for any geometric formula $\varphi(\mathbf{z})$ we have that

$$f^*(\{\mathbf{z} \ :\ \varphi(\mathbf{z})\}^{(M)}) = \{\mathbf{z} \ :\ \varphi(\mathbf{z})\}^{f^*(M)}$$

as subobjects of $Z_1^{f^*(M)} \times \ldots \times Z_n^{f^*(M)}$.

Proof:
We first show that the proposition holds for atomic formulae. f^* preserves small colimits and finite limits so $f^*\,(0_{\mathcal{E}})$ is initial in $\mathcal{F}$ and also $f^*(Z_1^{(M)} \times \ldots \times Z_n^{(M)}) = Z_1^{f^*(M)} \times \ldots \times Z_n^{f^*(M)}$ so f^* preserves the top and bottom elements of the subobject lattice of $Z_1^{f^*(M)} \times \ldots \times Z_n^{f^*(M)}$. In other notation, we have established that

$$f^*(\{\mathbf{z} \ :\ \top\}^{(M)}) = \{\mathbf{z} \ :\ \top\}^{f^*(M)}$$

and similarly for $\perp$.

Next notice that if $t_1, t_2 : Z_1 \times \ldots \times Z_n \to Y$ are terms then $\{\mathbf{z} \ :\ t_1 = t_2\}^{f^*(M)}$ is by definition the equaliser of the two arrows $t_1^{f^*(M)}, t_2^{f^*(M)}$ or equivalently, by the commuting square above, the equaliser of the two arrows $f^*(t_1^{(M)}), f^*(t_2^{(M)})$ and since f^* preserves equalisers we thus see

$$f^*(\{\mathbf{z} \ :\ t_1 = t_2\}^{(M)}) = \{\mathbf{z} \ :\ t_1 = t_2\}^{f^*(M)}$$

Similarly, f^* preserving pullbacks tells us that $f^*(\{\mathbf{z} \ :\ R(t_1, \ldots, t_k)\}^{(M)}) = \{\mathbf{z} \ :\ R(t_1, \ldots, t_k\}^{f^*(M)}$ for a relation $R \subseteq Y_1 \times \ldots \times Y_k$ and t_i with type Y_i.

Now we show the proposition holds for all geometric formulae. f^* preserves all finite limits and small colimits so immediately we see that if the proposition is known to hold for the formulae φ, ψ then it holds for $\varphi \wedge \psi$ and similarly if it is known to hold for φ_i with $i \in I$ then it must hold for $\bigvee_{i \in I} \varphi_i$.

Finally, notice that given f^* is a functor it preserves factorisations of arrows and given it preserves monomorphisms and epimorphisms we see it takes the image factorisation of an arrow $g : C \to D$ in $\mathcal{E}$ to a factorisation of f^*g into an epimorphism followed by a monomorphism. But a simple diagram chase shows that any epi-mono factorisation of an arrow is actually isomorphic to its image factorisation so for any subobject $M \rightarrowtail C$ we have that

$$f^*(\exists_g M) = \exists_{f^*g} f^*(C)$$

given the definition of $\exists_g : \mathrm{Sub}_{\mathcal{E}}(C) \to \mathrm{Sub}_{\mathcal{E}}(D)$ is by taking the image of $M \rightarrowtail C \xrightarrow{g} D$.

Thus by definition of the interpretation of a quantification $(\exists\, x \in X)$, if the proposition holds for a formula φ it must hold for $(\exists\, x \in X)\varphi$, finishing the proof. □

Then as claimed:

Theorem 4.1

If T is a geometric theory in a language $\mathcal{L}$, then a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ induces a functor $f^* : \mathrm{Mod}(T, \mathcal{E}) \to \mathrm{Mod}(T, \mathcal{F})$.

Proof:
Suppose M is an interpretation of $\mathcal{L}$ in $\mathcal{E}$ that makes the axiom

$$(\forall\, z_1 \in Z_1)(\forall\, z_2 \in Z_2)\dots(\forall\, z_n \in Z_n)(\varphi(\mathbf{z}) \implies \psi(\mathbf{z}))$$

valid for φ, ψ some geometric formulae. We need to show that

$$\{\,:\, (\forall\, z_1 \in Z_1)(\forall\, z_2 \in Z_2)\dots(\forall\, z_n \in Z_n)(\varphi(\mathbf{z}) \implies \psi(\mathbf{z}))\}^{f^*(M)} = 1$$

$$\iff \{\mathbf{z} \,:\, \varphi(\mathbf{z}) \implies \psi(\mathbf{z})\}^{f^*(M)} = Z_1^{f^*(M)} \times \dots \times Z_n^{f^*(M)}$$
$$\iff \{\mathbf{z} \,:\, \varphi(\mathbf{z})\}^{f^*(M)} \le \{\mathbf{z} \,:\, \psi(\mathbf{z})\}^{f^*(M)}$$

where the last equivalence is by the defining property of $\implies$ in Heyting algebras (the $\le$ is as subobjects of $Z_1^{f^*(M)} \times \dots \times Z_n^{f^*(M)}$). But we know that

$$\{\mathbf{z} \,:\, \varphi(\mathbf{z})\}^{(M)} \le \{\mathbf{z} \,:\, \psi(\mathbf{z})\}^{(M)}$$

in $\mathcal{E}$ and f^* is order preserving (as it is a functor) and that $f^*(\{\mathbf{z} \,:\, \varphi(\mathbf{z})\}^{(M)}) = \{\mathbf{z} \,:\, \varphi(\mathbf{z})\}^{f^*(M)}$ & similarly for ψ by **Proposition 4.1** so we are done. □

This solidifies one of Lawvere's maxims concerning categorical logic: *theories are functors*! Given this nice supply of functors on the category of Grothendieck topoi, we again ask the question (posed at the end of §1) *which of these are representable?* The answer is given by the next section.

5 Classifying Topoi

Remarkably, in a great culmination of logical and geometric intuition about Grothedieck topoi, every functor of the form $\mathrm{Mod}(T, -)$ is representable in the following sense.

Theorem 5.1 (*Classifying topoi*)

Let T be a geometric theory. Then there exists a topos $\mathcal{B}(T)$, called the **classifying topos** of T, such that for each cocomplete elementary topos $\mathcal{E}$ there is an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{B}(T)) \cong \mathrm{Mod}(T, \mathcal{E})$$

natural in $\mathcal{E}$, where $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{B}(T))$ is the category of geometric morphisms and geometric transformations between them (discussed at the end of §1).

Since Grothendieck topoi have all small colimits, this theorem works in particular for $\mathcal{E}$ being a Grothendieck topos. This section is devoted simply to describing the structure of the proof of this result; many of the details of the some of the propositions required are simply long calculations of little interest and so a full proof can be found [12] Chapter X.

The general idea of the proof is to construct a category $\mathbf{B}(T)$ called the **syntactic category** associated to the theory T. This is constructed in a way highly reminiscent of the construction of the Term Model in the proof of the Completeness Theorem for first-order logic: there we make the terms of the model be terms in the language $\mathcal{L}$ your theory is written in modulo equivalences that are provable from your theory T.

In our categorical context we similarly want to make the objects of our category be formulas $\varphi(\mathbf{z})$ (with input variables having type $Z = Z_1 \times \ldots \times Z_n$) but we haven't discussed any deduction systems or proof theory so we can't make our equivalence relation be "provably equal". One can imagine that if we had done so and proven soundness & completeness theorems, we would know that "provably equivalent" is the same thing as "equal in all models of T". Thus in fact our idea is to use exactly that as the equivalence relation, i.e. two geometric formulas φ, ψ yield the same equivalence class iff the subobjects

$$(\{\mathbf{z} \ : \ \varphi(\mathbf{z})\}^{(M)} \rightarrowtail Z_1^{(M)} \times \ldots \times Z_n^{(M)}) = (\{\mathbf{z}' \ : \ \psi(\mathbf{z}')\}^{(M)} \rightarrowtail Z_1^{(M)} \times \ldots \times Z_n^{(M)})$$

have equality for every interpretation M of the theory T ; we refer to such an equivalence class by the notation $[\varphi, Z]$.

The morphisms in this category should then be somehow terms or objects definable in our language $\mathcal{L}$. Well to any morphism $f : C \to D$ in a category we can associate its **graph** $\Gamma(f)$ which is the subobject of $A \times B$ given by

$$C \xrightarrow{\langle \mathrm{id}, f \rangle} C \times D$$

which is monic since $\mathrm{id} : C \to C$ is. Then morphisms $[\varphi, Z] \to [\psi, Y]$ in the category $\mathbf{B}(T)$ are geometric formulas $\sigma(\mathbf{z}, \mathbf{y})$ so that in every model M of T the subobject

$$\{(\mathbf{z}, \mathbf{y}) \ : \ \sigma(\mathbf{z}, \mathbf{y})\}^{(M)} \rightarrowtail Z^{(M)} \times Y^{(M)}$$

is the graph of an arrow $\{\mathbf{z} \ : \ \varphi(\mathbf{z})\}^{(M)} \to \{\mathbf{y} \ : \ \psi(\mathbf{y})\}^{(M)}$ modulo the equivalence relation of defining the same arrow in every model M of T. The idea is to prove that this is a category, i.e. composition is well-defined, and furthermore one with all finite limits and a Grothendieck topology $\mathcal{J}(T)$ given by a basis very similar to the epimorphic topology defined in **Example 1.5** in §1. Then finally $\mathcal{B}(T) := \mathrm{Sh}(\mathbf{B}(T), \mathcal{J}(T))$ and to show that this is indeed the classifying topos of T we call upon the following *Diaconescu-type* result:

5.1 The Pseudo-Diaconescu Theorem

Theorem 5.2

Let $(\mathcal{C}, J)$ be a site with finite limits and let $\mathcal{E}$ be a cocomplete elementary topos. Then there is an equivalence of categories

$$\underline{\mathrm{Hom}}\,(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J)) \cong \mathrm{ConLex}(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$, where $\mathrm{ConLex}(\mathcal{C}, \mathcal{E})$ is the category of all left exact functors $\mathcal{C} \to \mathcal{E}$ which carry covers in $(\mathcal{C}, J)$ to epimorphic families in $\mathcal{E}$ (functors with this last property are said to be **$\mathcal{J}$-continuous** or just **continuous**).

We'll start going through the above outline in more detail by first explaining something of where this result comes from. Let $\mathcal{C}$ now be any category. Given every element of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is a colimit of representables it stands to reason that a colimit-preserving functor $f : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$ should be entirely determined by its actions on the representables. In particular, using our result on this topic **Lemma 3.1** and the fact that f preserves colimits we have that

$$f(R) \cong \varinjlim \left(\int R \xrightarrow{\pi_R} \mathcal{C} \xrightarrow{y} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{f} \mathcal{E} \right)$$

for each presheaf $R : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. Given this observation, we might naturally think of taking functors $F : \mathcal{C} \to \mathcal{E}$ and extending their definition to be on all of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ by setting the image of the representables $\mathrm{Hom}(-, C) \mapsto F(C)$ to be that of F and trying to use the above construction:

Definition 5.1. Let $F : \mathcal{C} \to \mathcal{E}$ be a functor. Then the functor $(-) \otimes_{\mathcal{C}} F : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$ is defined by setting

$$R \otimes_{\mathcal{C}} F := \varinjlim \left(\int R \xrightarrow{\pi_R} \mathcal{C} \xrightarrow{F} \mathcal{E} \right)$$

for a presheaf $R : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and it is defined on natural transformations $\alpha : R \to S$ by noticing that they induce functors $\int \alpha : \int R \to \int S$ and thus a map $R \otimes_{\mathcal{C}} F \to S \otimes_{\mathcal{C}} F$ (by the universal property of colimits).

Similarly the functor $\underline{\mathrm{Hom}}_{\mathcal{E}}(F, -) : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is defined by setting

$$[\underline{\mathrm{Hom}}_{\mathcal{E}}(F, E)]\,(C) = \mathrm{Hom}_{\mathcal{E}}(F(C), E)$$

for an object E in $\mathcal{E}$ and works on morphisms in the natural way.

The functor $(-) \otimes_{\mathcal{C}} F$ is so named because of various properties it has that are similar to the traditional tensor product of modules over a ring. For example, it can then be checked that for any $F : \mathcal{C} \to \mathcal{E}$, the two induced functors have the adjoint relation $(-) \otimes_{\mathcal{C}} F \dashv \underline{\mathrm{Hom}}_{\mathcal{E}}(F, -)$, which is a kind of general form of the Tensor-Hom adjunction, and so $(-) \otimes_{\mathcal{C}} F$ always preserves small colimits. Inspired by this we then declare:

Definition 5.2. A functor $F : \mathcal{C} \to \mathcal{E}$ is said to be **flat** if its tensor functor $(-) \otimes_{\mathcal{C}} F$ is left exact. $\mathrm{Flat}(\mathcal{C}, \mathcal{E})$ is then the full subcategory of $[\mathcal{C}, \mathcal{E}]$ spanned by flat functors.

The traditional Diaconescu equivalence then states:

Theorem 5.3 (*Diaconescu*)

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ cocomplete. Then there is an equivalence of categories

$$\underline{\mathrm{Hom}}(\mathcal{E}, [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]) \cong \mathrm{Flat}(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

A complete proof of this and variants, including our **Pseudo-Diaconescu** result, can be found in [12] Chapter VII.

Proof:
Brief Sketch. In the direction $\mathrm{Flat}(\mathcal{C}, \mathcal{E}) \to \underline{\mathrm{Hom}}(\mathcal{E}, [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}])$, flat functors $A : \mathcal{C} \to \mathcal{E}$ are sent to the adjoint pair

$$A^*(-) := (-) \otimes_{\mathcal{C}} A \dashv \underline{\mathrm{Hom}}_{\mathcal{E}}(A, -) =: A_*(-)$$

which in fact make up a geometric morphism $\mathcal{E} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ given that the definition of flatness is that $(-) \otimes_{\mathcal{C}} A$ is left exact.

In the other direction, a geometric morphism $f : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is sent to the functor $f^* \circ y : \mathcal{C} \to \mathcal{E}$. But we've discussed above that for a presheaf $R : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$

$$R \otimes_{\mathcal{C}} (f^* \circ y) := \varinjlim \left(\int R \xrightarrow{\pi_R} \mathcal{C} \xrightarrow{y} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{f^*} \mathcal{E} \right) \cong f^*(R)$$

by **Lemma 3.1** and the fact that f^* preserves colimits so $(-) \otimes_{\mathcal{C}} (f^* \circ y) \cong f^*(-)$ which preserves finite limits, making $f^* \circ y : \mathcal{C} \to \mathcal{E}$ flat.

These two assignments $\underline{\mathrm{Hom}}(\mathcal{E}, [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]) \rightleftarrows \mathrm{Flat}(\mathcal{C}, \mathcal{E})$ can be then made into functors and with some work shown to define an equivalence between the two categories. ☐

A kind of "restriction" argument then shows an equivalence

$$\underline{\mathrm{Hom}}(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J)) \cong \mathrm{ConFlat}(\mathcal{C}, \mathcal{E})$$

for $(\mathcal{C}, J)$ a site and where ConFlat refers to the category of continuous flat functors. Finally, to establish the **Pseudo-Diaconescu Theorem** it's shown, via a mid-way concept of what it means for a functor to be *filtering*, that a functor $A : \mathcal{C} \to \mathcal{E}$ is flat iff it is left exact assuming $\mathcal{C}$ has all finite limits.

Pseudo-Diaconescu tells us that once we construct our category $\mathbf{B}(T)$ then if we show it has all finite limits we know that

$$\underline{\text{Hom}}(\mathcal{E}, \mathcal{B}(T)) \cong \text{ConLex}(\mathbf{B}(T), \mathcal{E})$$

naturally in cocomplete topoi $\mathcal{E}$ meaning that to prove **Theorem 5.1** it is sufficient to prove a natural equivalence

$$\text{Mod}(T, \mathcal{E}) \cong \text{ConLex}(\mathbf{B}(T), \mathcal{E})$$

Our first goal is thus to properly define $\mathbf{B}(T)$, show it is a category and has all finite limits.

5.2 Definable Objects

In order to construct $\mathbf{B}(T)$ and proceed with the equivalence mentioned in the previous section, it is easiest to look at the various "shadows" of $\mathbf{B}(T)$ given by the category of *definable objects* in a Grothendieck topos $\mathcal{E}$ given a model M of a geometric theory T.

Definition 5.3. Let $\mathcal{E}$ be a Grothendieck topos, T be a geometric theory and M a model of T in $\mathcal{E}$. Then the category $\mathbf{Def}(M)$ has as objects pairs (A, X) where $X = (X_1, \ldots, X_n)$ is a list of types and $A \rightarrowtail X_1^{(M)} \times \ldots \times X_n^{(M)}$ is a **definable subobject**, i.e. such that there is some geometric formula $\varphi(\mathbf{x})$ such that

$$\{\mathbf{x} \ : \ \varphi(\mathbf{x})\}^{(M)} = A$$

as subobjects of $X_1^{(M)} \times \ldots \times X_n^{(M)}$ and it has arrows $s : (A, X) \to (B, Y)$ being morphisms $s : A \to B$ in $\mathcal{E}$ such that there is a geometric formula $\sigma(\mathbf{x}, \mathbf{y})$ with

$$\{\sigma(\mathbf{x}, \mathbf{y}) \ : \ (\mathbf{x}, \mathbf{y})\}^{(M)} \rightarrowtail X_1^{(M)} \times \ldots \times X_n^{(M)} \times Y_1^{(M)} \times \ldots \times X_m^{(M)}$$

giving the same subobject as the graph $\Gamma(s) \rightarrowtail A \times B$ viewed as a subobject of $X_1^{(M)} \times \ldots \times X_n^{(M)} \times Y_1^{(M)} \times \ldots \times Y_m^{(M)}$.

From now on we'll refer to $X_1^{(M)} \times \ldots \times X_n^{(M)}$ as simply $X^{(M)}$. Then we show that $\mathbf{Def}(M)$ is a category:

Proposition 5.1

$\mathbf{Def}(M)$ is a category in the sense that it has identity arrows for each object and composition is well-defined, i.e. the composition of two definable arrows is definable.

Proof:
For any object (A, X) in $\mathbf{Def}(M)$, we claim that $\text{id} : (A, X) \to (A, X)$ chosen to be the underlying arrow $\text{id} : A \to A$ in $\mathcal{E}$ works. Well $\Gamma(\text{id})$ is just the diagonal subobject $\Delta_A \rightarrowtail A \times A$ and, if $\varphi(\mathbf{x})$ is the defining formula of A, then we can calculate the subobject of $X^{(M)} \times X^{(M)}$

$$\{(\mathbf{x}, \mathbf{x}') \ : \ \varphi(\mathbf{x}) \wedge \varphi(\mathbf{x}') \wedge x_1 = x_1' \wedge \ldots \wedge x_n = x_n'\}^{(M)}$$

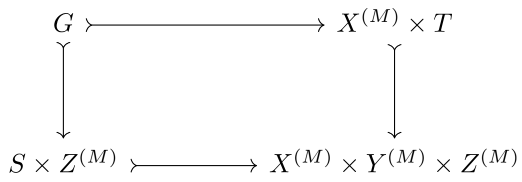
$$= \{(\mathbf{x}, \mathbf{x}') \ : \ \varphi(\mathbf{x})\}^{(M)} \wedge \{(\mathbf{x}, \mathbf{x}') \ : \ \varphi(\mathbf{x}')\}^{(M)} \wedge \Delta_X = (A \times X^{(M)}) \wedge (X^{(M)} \times A) \wedge \Delta_X$$

which is just Δ_A so the map $\text{id} : A \to A$ is indeed definable.

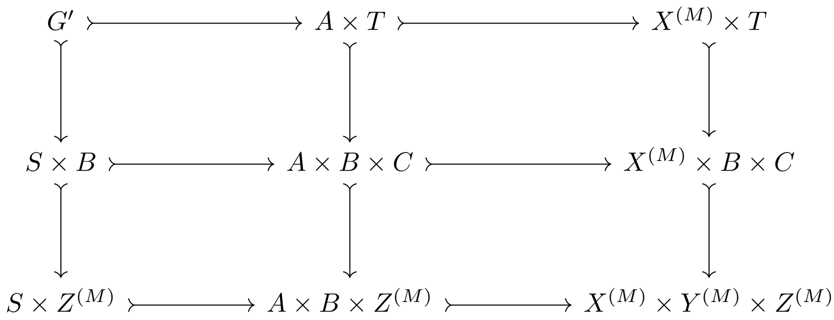
Now suppose $s : (A, X) \to (B, Y), t : (B, Y) \to (C, Z)$ are definable with $\sigma(\mathbf{x}, \mathbf{y}), \tau(\mathbf{y}, \mathbf{z})$ defining the graphs of s, t respectively. The natural guess then is that the graph of $t \circ s : A \to C$ should be

$$R := \{(\mathbf{x}, \mathbf{z}) \ : \ (\exists\, \mathbf{y} \in Y)(\sigma(\mathbf{x}, \mathbf{y}) \wedge \tau(\mathbf{y}, \mathbf{z})\}^{(M)}$$

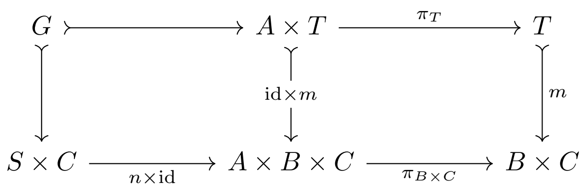
which we now endeavour to prove (note that this formula is geometric if σ, τ are, as needed). Let's first consider the subobject of $X^{(M)} \times Y^{(M)} \times Z^{(M)}$ given by $G := \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \ : \ \sigma(\mathbf{x}, \mathbf{y}) \wedge \tau(\mathbf{y}, \mathbf{z})\}^{(M)}$ and calling $S := \{(\mathbf{x}, \mathbf{y}) \ : \ \sigma(\mathbf{x}, \mathbf{y})\}^{(M)}, T := \{(\mathbf{y}, \mathbf{z}) \ : \ \tau(\mathbf{y}, \mathbf{z})\}$ then it fits into the left pullback square



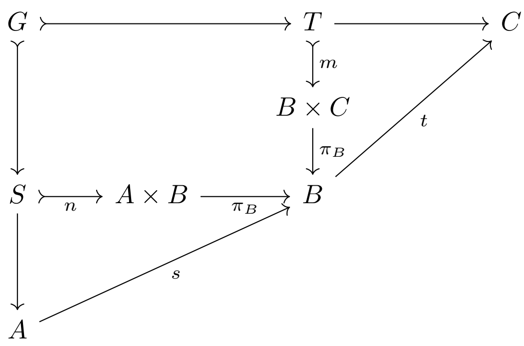
and now consider



We can note that the outermost square is the previous pullback diagram so that $G' \cong G$ and then note that all the squares other than the top-left are pullbacks by inspection. Thus the top-left square is a pullback. Then consider



where we've named the subobject arrows as $m : T \rightarrowtail B \times C, n : S \rightarrowtail A \times B$. Given the leftmost square is a pullback by our above work and the rightmost square is a pullback by inspection we get that the outermost rectangle is a pullback and by the same argumentation we get the square in



is a pullback, where the arrow $S \to A$ is the composite $S \xrightarrow{n} A \times B \xrightarrow{\pi_A} A$ which is an isomorphism given S is the subobject $\Gamma(s) \to A \times B$. Similarly the composite arrow $T \xrightarrow{m} B \times C \xrightarrow{\pi_B} B$ is an isomorphism. Since pullbacks preserve epis and monos and $\mathcal{E}$ is balanced, we see that the arrow $G \to S$ is an isomorphism also.

But then since the outermost triangle commutes, its leftmost side is an isomorphism and the hypotenuse is $t \circ s$ we see that the two arrows $G \to A, G \to C$ make G into the graph $\Gamma(t \circ s)$ subobject of $A \times C$.

Finally recall that by definition of our interpretation of existential quantifiers,

$$\{(\mathbf{x},\mathbf{z})\ :\ (\exists\, \mathbf{y}\in Y)\sigma(\mathbf{x},\mathbf{y})\wedge\tau(\mathbf{y},\mathbf{z})\}^{(M)}$$

is the image of G under the projection $\pi_{A\times C}$ but we just saw that this composite $G\to A\times C$ is monic so indeed $G\cong R$, showing that the above formula does define the graph of $t\circ s$. □

We worked out the above proof in full but it was fiddly. All other proofs about definable objects involve similar diagrammatic computations with pullbacks but can worked out by following your intuition about Set; we will not be so detailed in subsequent proofs.

Proposition 5.2

The category $\mathbf{Def}(M)$ has all finite limits.

Proof:
It is sufficient to show it has a terminal object and all pullbacks. This is simply because we can consider the object $(1,1)$, i.e. the subobject 1 of the empty product of types 1 which is defined as the top subobject $\{\ :\ \top\}^{(M)}$. Any other definable object (A,X) has a unique arrow on the underlying object $A\to 1$ and this is definable as the graph is isomorphic to the subobject A itself.

To check it has pullbacks, consider a pair of arrows $(A,X)\xrightarrow{s}(C,Z)\xleftarrow{t}(B,Y)$. It can then be checked that

$$\begin{array}{ccc}(A\times_C B,(X,Y)) & \longrightarrow & (B,Y)\\ \downarrow & & \downarrow{\scriptstyle t}\\ (A,X) & \xrightarrow[s]{} & (C,Z)\end{array}$$

is a pullback square in $\mathbf{Def}(M)$, where (X,Y) is the concatenation of the lists X,Y of types, i.e. that

- $A\times_C B$ is a definable subobject of $X^{(M)}\times Y^{(M)}$, in particular by the intuitive definition $\{(\mathbf{x},\mathbf{y})\ :\ (\exists\,\mathbf{z}\in Z)\sigma(\mathbf{x},\mathbf{z})\wedge\tau(\mathbf{y},\mathbf{z})\}^{(M)}$ which uses a geometric formula assuming that σ,τ are.
- The projection arrows $A\times_C B\to B,A$ are definable.
- The square is in fact a pullback in $\mathbf{Def}(M)$ (rather than just one in the underlying category $\mathcal{E}$).

□

These proofs have in fact given slightly more. Clearly the category comes with a forgetful functor $\mathbf{Def}(M)\to\mathcal{E}$ sending $(A,X)\mapsto A$ and arrows $(A,X)\to(B,Y)$ to their underlying arrows $A\to B$ and the proof above shows that this functor preserves all limits.

It also inherits from $\mathcal{E}$ the epimorphic topology discussed in **Example 1.5** in §1 where a finite family $(f_i : C_i\to C)$ is a cover of C iff the coproduct map $f:\bigsqcup_{i=1}^n C_i\to C$ is an epimorphism, i.e. a finite family $(f_i:(C_i,X_i)\to(C,X))$ in $\mathbf{Def}(M)$ is considered a cover iff the underlying family in $\mathcal{E}$ is.

Proposition 5.3

The epimorphic topology on $\mathbf{Def}(M)$ is **subcanonical**, i.e. all representable functors $\mathrm{Hom}(-,(B,Y))$ are sheaves with this topology.

Proof:
To show this, it is sufficient to show that if $\{f_i:(C_i,X_i)\to(C,X)\}$ is a cover and we have a matching family on this cover, i.e. a member $c_i\in\mathrm{Hom}((C_i,X_i),(B,Y))$ for each $1\le i\le n$

such that the outer square in any diagram of the form

$$\begin{array}{ccccc} (C_i \times_C C_j, (X_i, X_j)) & \longrightarrow & (C_j, X_j) & & \\ \downarrow & & \downarrow f_j & \searrow c_j & \\ (C_i, X_i) & \xrightarrow{f_i} & (C, X) & & \\ & \searrow_{c_i} & & & (B, Y) \end{array}$$

commutes (as precomposition of c_i by the map $C_i \times_C C_j \to C_i$ corresponds to the restriction $c_i|_{C_i \times_C C_j}$), then it has a unique $c \in \mathrm{Hom}((C,X),(B,Y))$ which has $c|_{f_i} = c \circ f_i = c_i$.

Now we know that the coproduct $f : \bigsqcup_{i=1}^n C_i \to C$ of the f_i is an epimorphism and work along the lines of what was done in §2 can show that in any elementary topos every epimorphism is a coequaliser and specifically the coequaliser of its kernel pair, i.e. the morphisms k_1, k_2 such that

$$\begin{array}{ccc} (\bigsqcup_{i=1}^n C_i) \times_C (\bigsqcup_{j=1}^n C_j) & \xrightarrow[k_2]{} & \bigsqcup_{i=1}^n C_i \\ \downarrow k_1 & & \downarrow f \\ \bigsqcup_{i=1}^n C_i & \xrightarrow[f]{} & C \end{array}$$

is a pullback. Since we know from **Proposition 2.4** that pullbacks in topoi preserve small colimits, they must in particular preserve coproducts and so, applying this first for i then for j, we see that we have a coequaliser diagram

$$\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n C_i \times_C C_j \rightrightarrows_{k_2}^{k_1} \bigsqcup_{i=1}^n C_i \xrightarrow{f} C$$

But the condition above on the $c_i : C_i \to B$ then tells us that the coproduct map $c' : \bigsqcup_{i=1}^n C_i \to B$ equalises these arrows $\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n C_i \times_C C_j \rightrightarrows \bigsqcup_{i=1}^n C_i$ so that c' factors as $c \circ f$ for some map $c : C \to B$. Precomposing the equation $c' = c \circ f$ by the inclusions $C_k \to \bigsqcup_{i=1}^n C_i$ tells us that indeed $c \circ f_k = c_k$ for each k.

It can then be checked that this map $c : C \to B$ is actually definable so that it defines a map $c : (C,X) \to (B,Y)$. The formula is again what you'd expect if you think about $\mathcal{E}$ as if it was the category of sets: if c_i, f_i, C_i, C are defined by formulas $\kappa_i(\mathbf{x}^i, \mathbf{y}), \tau_i(\mathbf{x}^i, \mathbf{x}), \varphi_i(\mathbf{x}^i), \psi(\mathbf{x})$ respectively then c is defined by the (geometric) formula

$$\bigvee_{i=1}^n (\exists\, \mathbf{x}^i \in X_i)(\kappa_i(\mathbf{x}^i, \mathbf{y}) \wedge \tau_i(\mathbf{x}^i, \mathbf{x}) \wedge \varphi_i(\mathbf{x}^i) \wedge \psi(\mathbf{x}))$$

□

5.3 The Category $\mathbf{B}(T)$

We now recap the construction of the category $\mathbf{B}(T)$, prove it *is* a category and show it has the properties we desired.

Definition 5.4. The category $\mathbf{B}(T)$ has as objects $[\varphi, Z]$, i.e. equivalence classes of pairs of geometric formulas φ and tuples of types $Z = (Z_1, \ldots, Z_n)$ where $(\varphi, Z), (\psi, Z)$ are equivalent iff the subobjects

$$(\{\mathbf{z} \ : \ \varphi(\mathbf{z})\}^{(M)} \rightarrowtail Z_1^{(M)} \times \ldots \times Z_n^{(M)}) = (\{\mathbf{z}' \ : \ \psi(\mathbf{z}')\}^{(M)} \rightarrowtail Z_1^{(M)} \times \ldots \times Z_n^{(M)})$$

are equal for every interpretation M of the theory T, and has as morphisms $[\varphi, Z] \to [\psi, Y]$ equivalence classes of geometric formulas $\sigma(\mathbf{z}, \mathbf{y})$ which in every model M of T define the graph of a morphism between the objects defined by φ, ψ (where again two such formulas are equivalent iff they define the same graph in every model M).

Then:

Theorem 5.4

We have:
1. $\mathbf{B}(T)$ is a well-defined category.
2. $\mathbf{B}(T)$ has all finite limits and the "forgetful" functor $F_M : \mathbf{B}(T) \to \mathbf{Def}(M)$ preserves these limits.
3. $\mathbf{B}(T)$ inherits a Grothendieck topology from the categories $\mathbf{Def}(M)$ which is subcanonical and such that $F_M : \mathbf{B}(T) \to \mathbf{Def}(M)$ is continuous.

$F_M : \mathbf{B}(T) \to \mathbf{Def}(M)$ works exactly as you'd imagine: it sends objects $[\varphi, Z]$ to (A, Z) where $A \rightarrowtail Z^{(M)}$ is the subobject equal to the interpretation of φ in M and it sends arrows in $\mathbf{B}(T)$ similarly to the arrows they define in $\mathbf{Def}(M)$.

Proof:

Everything here is a simple consequence of the proofs given in the previous section.

1. The identity arrow $\text{id} : [\varphi, X] \to [\varphi, X]$ is represented by the formula

$$\varphi(\mathbf{x}) \wedge \varphi(\mathbf{x}') \wedge (x_1 = x_1') \wedge \ldots \wedge (x_n = x_n')$$

   as this is the graph of the identity arrow of $\{\mathbf{x} \ : \ \varphi(\mathbf{x})\}^{(M)}$ for any model M of T in a topos $\mathcal{E}$ as shown in the proof of **Proposition 5.1**. Similarly if $s : [\varphi, X] \to [\psi, Y], t : [\psi, Y] \to [\zeta, Z]$ define arrows in $\mathbf{B}(T)$ are arrows represented by the geometric formulas $\sigma(\mathbf{x}, \mathbf{y}), \tau(\mathbf{y}, \mathbf{z})$ then their composition $t \circ s : [\varphi, X] \to [\zeta, Z]$ was shown to be represented by the geometric formula $(\exists\, \mathbf{y} \in Y)(\sigma(\mathbf{x}, \mathbf{y}) \wedge \tau(\mathbf{y}, \mathbf{z}))$ in every model, so composition is well-defined in $\mathbf{B}(T)$.

2. We showed not only that all finite limits exist in each category $\mathbf{Def}(M)$ in **Proposition 5.2** but that they are each definable by geometric formulas (which work regardless of the choice of model M and category $\mathcal{E}$) so these formulas represent the limits in $\mathbf{B}(T)$ and show they are preserved by $F_M : \mathbf{B}(T) \to \mathbf{Def}(M)$.

3. The topology $J(T)$ is defined by giving a basis where $\{f_i : [\varphi_i, X_i] \to [\varphi, X]\}$ is said to be a cover iff for all models M, the functor F_M sends this family to a cover in $\mathbf{Def}(M)$; this is easily checked to be a basis and makes each F_M continuous by definition. It can then be checked that **Proposition 5.3**'s statement that the topology on each $\mathbf{Def}(M)$ being subcanonical implies $J(T)$ is subcanonical on $\mathbf{B}(T)$.

□

Finally we prove the equivalence of categories

$$\text{Mod}(T, \mathcal{E}) \cong \text{ConLex}(\mathbf{B}(T), \mathcal{E})$$

for each cocomplete topos $\mathcal{E}$ and geometric theory T. As discussed above, **Theorem 5.4** and Pseudo-Diaconescu would then tell us that

$$\underline{\text{Hom}}(\mathcal{E}, \mathcal{B}(T)) \cong \text{ConLex}(\mathbf{B}(T), \mathcal{E}) \cong \text{Mod}(T, \mathcal{E})$$

completing at last the proof of **Theorem 5.1**.

Proof:
To prove the equivalence, we will exhibit a functor that takes a model M of T in the topos $\mathcal{E}$ to a continuous left exact functor $A_M : \mathbf{B}(T) \to \mathcal{E}$ and then another functor that takes continuous left exact functors $A : \mathbf{B}(T) \to \mathcal{E}$ to models M_A of T in $\mathcal{E}$. We will then quickly observe that they are inverse up to natural isomorphism and be done.

Let M be a model of T in $\mathcal{E}$. Then we can define $A_M : \mathbf{B}(T) \to \mathcal{E}$ to be the composite functor $\mathbf{B}(T) \xrightarrow{F_M} \mathbf{Def}(M) \to \mathcal{E}$ where the second functor is the forgetful functor discussed in §5.2. We know from **Theorem 5.4**, **Proposition 5.2** and the definitions of the Grothendieck topologies on $\mathbf{B}(T), \mathbf{Def}(M), \mathcal{E}$ that A_M as defined is left exact and continuous. If $H : M \to M'$ is a homomorphism between interpretations M, M' of T in $\mathcal{E}$ then an induction argument on the construction of formulas φ shows that this induces arrows between all subobjects

$$\begin{array}{ccc}
\{\mathbf{x} : \phi(\mathbf{x})\}^{(M)} & \rightarrowtail & X_1^{(M)} \times \ldots \times X_n^{(M)} \\
\downarrow & & \downarrow {\scriptstyle H_{X_1} \times \ldots \times H_{X_n}} \\
\{\mathbf{x} : \phi(\mathbf{x})\}^{(M')} & \rightarrowtail & X_1^{(M')} \times \ldots \times X_n^{(M')}
\end{array}$$

defined by those interpretations in the sense that the right-down composite arrow $\{\mathbf{x} : \varphi(\mathbf{x})\}^{(M)} \to X_1^{(M')} \times \ldots \times X_n^{(M')}$ factors through $\{\mathbf{x} : \varphi(\mathbf{x})\}^{(M')}$. These can be checked to together form a natural transformation $A_M \implies A_{M'}$ thus making the assignment $M \mapsto A_M$ functorial.

In the other direction, suppose $A : \mathbf{B}(T) \to \mathcal{E}$ is a continuous left exact functor. We can define an interpretation M_A of the language $\mathcal{L}$ is written by:

1. Assigning objects as $X^{(M_A)} := A([\top, X])$ for each simple type X in $\mathcal{L}$, as this is the obvious way of referring to X itself using a formula.
2. Each relation $R \subseteq Z_1 \times \ldots \times Z_n$ is given the object $R^{(M_A)} := A([R(\mathbf{z}), Z])$. $[R, X]$ has a monomorphism into $[\top, Z]$ in $\mathbf{B}(T)$ since its interpretations $\{\mathbf{z} : R(\mathbf{z})\}^{(M)}$ are subobjects of $Z_1^{(M)} \times \ldots \times Z_n^{(M)}$ for every model M and limits in $\mathbf{B}(T)$ are constructed from those in $\mathbf{Def}(M)$ for all M. Since $A : \mathbf{B}(T) \to \mathcal{E}$ is left exact we see it sends this subobject arrow to an arrow
$$R^{(M_A)} := A([R, Z]) \rightarrowtail A([\top, Z]) = A([\top, Z_1] \times \ldots \times [\top, Z_n]) \cong Z_1^{(M_A)} \times \ldots \times Z_n^{(M_A)}$$
making $R^{(M_A)}$ into a subobject of $Z_1^{(M_A)} \times \ldots \times Z_n^{(M_A)}$ as needed.
3. Each function symbol $f : Z_1 \times \ldots \times Z_n \to Y$ has a corresponding arrow $f : [\top, Z] \to [\top, Y]$ in $\mathbf{B}(T)$ since in any model its interpretation's graph is defined by the atomic formula $f(\mathbf{z}) = \mathbf{y}$ and then $f^{(M_A)} := A(f)$.

It can then be checked that this interpretation M_A has for every geometric formula φ

$$\{\mathbf{x} : \varphi(\mathbf{x})\}^{(M_A)} = A([\varphi, X])$$

by induction on the complexity of φ, where left exactness is used for the $\wedge, =$ parts while continuity is used for the $\vee, \exists$ portions (since the definitions of their interpretations both make reference to epimorphisms). Finally notice that this assignment $A \mapsto M_A$ is functorial as is evident from the construction of the interpretation M_A above.

Let's now check that these functors form an equivalence. Suppose M is a model of T in $\mathcal{E}$. Then $M \mapsto A_M \mapsto M_{A_M}$ by the functors and we wish to show that $M \cong M_{A_M}$ naturally in M. Well for any sort X of the language $\mathcal{L}$ we have

$$X^{(M_{A_M})} := A_M([\top, X]) := \{\mathbf{x} : \top\}^{(M)} := X$$

all by the definitions of the functors and the interpretation of $\top$. Similarly for any relation $R \subseteq X_1 \times \ldots \times X_n$ in $\mathcal{L}$ we have

$$R^{(M_{A_M})} := A_M([R, X]) := \{\mathbf{x} \,:\, R(\mathbf{x})\}^{(M)} =: R^{(M)}$$

and the same computation can be done for all function symbols of $\mathcal{L}$. Thus clearly $M \cong M_{A_M}$ simply by unravelling all the definitions.

In the other direction, suppose $A : \mathbf{B}(T) \to \mathcal{E}$ is left exact and continuous. This is sent $A \mapsto M_A \mapsto A_{M_A}$ by the functors and this time the check is even simpler: for any object $[\varphi, X]$ in $\mathbf{B}(T)$ we just have

$$A_{M_A}([\varphi, X]) := \{\mathbf{x} \,:\, \varphi(\mathbf{x})\}^{(M_A)} \cong A([\varphi, X])$$

so that $A_{M_A} \cong A$. Thus we are done! □

6 Examples of Classifiers

Now that we have an incredibly general result telling us that any geometric theory we desire has a classifying topos, it might be of interest to compute two such topoi explicitly and one more is stated at the end without proof.

Example 6.1 (Object Classifier)**.** The simplest theory to look at is the *theory of objects*: there is one sort X, equality and no function symbols, relation symbols or axioms of any kind. A model of this theory is simply an object in a category $\mathcal{E}$ and a homomorphism between such models is simply a morphism in $\mathcal{E}$ between the assigned objects. In other words, we should have an equivalence of categories

$$\mathcal{E} \cong \mathrm{Mod}(T, \mathcal{E}) \cong \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{B}(T))$$

for the classifying topos $\mathcal{B}(T)$ of this theory. We claim that you can take $\mathcal{B}(T) = [\mathrm{FinSet}, \mathrm{Set}]$ where FinSet is the category of all finite sets and functions between them. To prove this we apply **Diaconescu's Theorem** as discussed above and the discussion following it to see that

$$\underline{\mathrm{Hom}}(\mathcal{E}, [\mathrm{FinSet}, \mathrm{Set}]) \cong \mathrm{Flat}(\mathrm{FinSet}^{\mathrm{op}}, \mathcal{E}) = \mathrm{Lex}(\mathrm{FinSet}^{\mathrm{op}}, \mathcal{E})$$

It is therefore sufficient to prove that there is an equivalence of categories

$$\mathrm{ev}_{\{*\}} : \mathrm{Rex}(\mathrm{FinSet}, \mathcal{E}) \to \mathcal{E}$$

where Rex refers to right exact functors, i.e. those which preserve finite colimits, and $\mathcal{E}$ is assumed to have all finite colimits since it is easier to reason about FinSet than $\mathrm{FinSet}^{\mathrm{op}}$.

As suggested by the notation, the equivalence $\mathrm{ev}_{\{*\}}$ will be given in one direction by evaluating a right exact functor $F : \mathrm{FinSet} \to \mathcal{E}$ at a specified singleton $\{*\}$. In the other direction, let E be an object of $\mathcal{E}$ and let's define functor $\varphi_E : \mathrm{FinSet} \to \mathcal{E}$ on objects by replacing each element of a set by a distinct copy of E, i.e. $\varphi_E(S) = \coprod_{s \in S} E$ for a set S, and similarly sending functions $f : S \to T$ to their "E-inflated versions", i.e. to the unique arrow making

$$\begin{array}{ccc} \coprod_{s\in S} E & \xrightarrow{\varphi_E(f)} & \coprod_{t\in T} E \\ \uparrow{\scriptstyle s_0} & & \uparrow{\scriptstyle f(s_0)} \\ E & \xrightarrow{\mathrm{id}} & E \end{array}$$

commute, where the arrows $E \xrightarrow{s_0} \coprod_{s \in S} E$ are the coproduct inclusions corresponding to the element $s_0 \in S$. φ_E preserves all coproducts in FinSet and, since every set in FinSet is a finite coproduct of the singleton $\{*\}$, it can be checked that this is sufficient for φ_E to be right exact.

These two functors $\mathrm{ev}_{\{*\}}, \varphi_{(-)}$ are clearly equivalences since if E is an object in $\mathcal{E}$ it is sent $E \mapsto \varphi_E \mapsto \mathrm{ev}_{\{*\}}(\varphi_E) := \varphi_E(\{*\}) = E$ and on the other hand if $F : \mathrm{FinSet} \to \mathcal{E}$ is right exact then

$$F \mapsto \mathrm{ev}_{\{*\}}(F) := F(\{*\}) \mapsto \varphi_{F(\{*\})}$$

but

$$\varphi_{F(\{*\})}(S) := \bigsqcup_{s\in S} F(\{*\}) \cong F\left(\bigsqcup_{s\in S}\{*\}\right) \cong F(S)$$

so clearly $F \cong \varphi_{F(\{*\})}$ as required.

This is part of a more general theorem for *theories of presheaf type*, i.e. those whose classifying topoi are a presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. We say that an object C in a category $\mathcal{C}$ is **compact** iff $\mathrm{Hom}(C, -)$ preserves filtered colimits and denote by $\mathcal{C}_c$ the full subcategory of compact objects. Such objects are often somewhat "small" in categories, e.g. the compact objects in the category of groups are precisely the finitely presentable groups.

Theorem 6.1

Let T be a geometric theory of presheaf type. Then $[\mathrm{Mod}(T, \mathrm{Set})_c, \mathrm{Set}]$ is a classifying topos for T.

A proof of this along with simple characterisations and sufficient conditions for a theory to be of presheaf type can be found in [5], Chapter 6. Indeed the theorem can be applied to the first part of the next example.

Example 6.2 (Rings)**.** We are now interested in finding a classifying topos for the theory of rings and looking at the above theorem (though not knowing if this theory is of presheaf-type) we might guess that it is $[\mathrm{FinRing}, \mathrm{Set}]$ where FinRing is the category of finitely presented rings over $\mathbb{Z}$, i.e. those isomorphic to quotients of polynomial rings over $\mathbb{Z}$ by finitely generated ideals. To show this, as in the previous example Diaconescu tells you that it is sufficient to show an equivalence

$$\mathrm{ev}_{\mathbb{Z}[X]} : \mathrm{Lex}(\mathrm{FinRing}^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Ring}(\mathcal{E})$$

for each cocomplete topos $\mathcal{E}$ where $\mathrm{Ring}(\mathcal{E})$ denotes the category of models of the theory of rings in $\mathcal{E}$. Again as the notation suggests the Lex $\to$ Ring direction takes a left exact functor $F : \mathrm{FinRing}^{\mathrm{op}} \to \mathcal{E}$ and evaluates it at $\mathbb{Z}[X]$.

Let's first think about $\mathrm{FinRing}^{\mathrm{op}}$. FinRing has all finite coproducts since they are just tensor products of rings over $\mathbb{Z}$ and it also has coequalisers of pairs of maps since

$$\mathbb{Z}[X_1, \ldots, X_n]/\langle Y_1, \ldots, Y_m\rangle \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathbb{Z}[W_1, \ldots, W_k]/\langle T_1, \ldots, T_l\rangle$$

has coequaliser $\mathbb{Z}[W_1, \ldots, W_k]/\left\langle T_1, \ldots, T_l, \varphi(X_1) - \psi(X_1), \ldots, \varphi(X_n) - \psi(X_n)\right\rangle$ equipped with the quotient map from the rightmost ring. Thus FinRing has all finite colimits and so $\mathrm{FinRing}^{\mathrm{op}}$ has all finite limits. Now the evaluation map $\mathrm{ev}_{\mathbb{Z}[X]}$ isn't even well-defined unless $\mathbb{Z}[X]$ is a ring object in $\mathrm{FinRing}^{\mathrm{op}}$ and F carries it to one in $\mathcal{E}$. Well if it is a ring object in the domain, then considering the fact that the theory of rings is a finite list of axioms simply consisting of equalities of various algebraic expressions it can be unravelled to be the requirement that an object R comes equipped with arrows $0_R, 1_R : 1 \to R$, $- : R \to R$ and $\cdot, + : R \times R \to R$ such that various diagrams commute and are equalisers and since a left exact functor preserves these structures, $F(\mathbb{Z}[X])$ will be a ring object in $\mathcal{E}$; in other words, we can look at the parts of **Proposition 4.1, Theorem 4.1** which only require the functor in question to be left-exact.

Why is $\mathbb{Z}[X]$ still a ring object in the opposite category $\mathrm{FinRing}^{\mathrm{op}}$? Well we claim that its zero and unit arrows are the opposites of $\mathrm{ev}_0, \mathrm{ev}_1 : \mathbb{Z}[X] \to \mathbb{Z}$ which evaluate polynomials at $0, 1 \in \mathbb{Z}$, its negation is the opposite map to $\mathbb{Z}[X] \xrightarrow{-} \mathbb{Z}[X]$ and addition and multiplication are the opposites to the maps $\mathbb{Z}[X] \xrightarrow{X\mapsto X_1+X_2} \mathbb{Z}[X_1,X_2], \mathbb{Z}[X] \xrightarrow{X\mapsto X_1X_2} \mathbb{Z}[X_1,X_2]$ (using the fact that $\mathbb{Z}[X]\otimes\mathbb{Z}[X] \cong \mathbb{Z}[X_1,X_2]$). These maps can then be shown to make $\mathbb{Z}[X]$ into a ring in the opposite category!

For example, the axiom stating the associativity of addition for a ring object R says that the subobject of $R\times R\times R$ for which the operation $+ : R\times R \to R$ has the expressions $(x+y)+z, x+(y+z)$ agreeing is the whole thing, i.e. that the diagram

$$\begin{array}{ccccc} (R\times R)\times R & & \xrightarrow{\cong} & & R\times(R\times R) \\ \downarrow{\scriptstyle (+)\times\mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}\times(+)} \\ R\times R & \xrightarrow{+} & R & \xleftarrow{+} & R\times R \end{array}$$

commutes. Checking this for $\mathbb{Z}[X]$ in $\mathrm{FinRing}^{\mathrm{op}}$ amounts to

$$\begin{array}{ccc} \mathbb{Z}[X_1,X_2,X_3] & \xleftarrow{X'\mapsto X_2+X_3} & \mathbb{Z}[X_1,X'] \\ \uparrow{\scriptstyle X'\mapsto X_1+X_2} & & \uparrow{\scriptstyle X\mapsto X_1+X'} \\ \mathbb{Z}[X',X_3] & \xleftarrow{X'+X_3\leftarrow X} & \mathbb{Z}[X] \end{array}$$

which can be easily checked to commute. Another way of thinking about this is that as well as being a ring, $\mathbb{Z}[X]$ is additionally a *coring* and the above showed that its coaddition operation is coassociative. It is easy to then check that natural transformations between functors $F \implies G$ yield ring homomorphisms $F(\mathbb{Z}[X]) \to G(\mathbb{Z}[X])$ so that $\mathrm{ev}_{\mathbb{Z}[X]}$ is a functor $\mathrm{Lex}(\mathrm{FinRing}^{\mathrm{op}},\mathcal{E}) \to \mathrm{Ring}(\mathcal{E})$.

In the other direction, we start with a ring object R in $\mathcal{E}$ and wish to construct a left exact functor $\varphi_R : \mathrm{FinRing}^{\mathrm{op}} \to \mathcal{E}$. This is done in stages of increasing complexity:

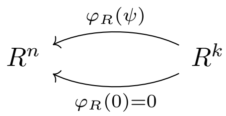
1. Given we want the assignment $R \mapsto \varphi_R$ to be inverse to the evaluation functor above, we must define $\varphi_R(\mathbb{Z}[X]) = R$.
2. Since we want φ_R to be left exact and in particular preserve products, we may as well define $\varphi_R(\mathbb{Z}[X_1,\ldots,X_n]) := R^n$ (given $Z[X_1,\ldots,X_n] \cong \mathbb{Z}[X]^{\otimes n}$).
3. We now define φ_R on *certain* morphisms: those of the form $\mathbb{Z}[X_1,\ldots,X_n] \to \mathbb{Z}[W_1,\ldots,W_k]$ which should produce a map $R^k \to R^n$ in $\mathcal{E}$. Well the ring map is defined by the images of $X_1,\ldots,X_n$ which are polynomials $P_1,\ldots,P_n \in \mathbb{Z}[W_1,\ldots,W_k]$ and these each define a map $R^k \to R$ as the polynomial can be written as a map using the arrows $\cdot, + : R\times R \to R$ etc. To illustrate, one can write the polynomial $x^2 \in \mathbb{Z}[X]$ as the composite arrow $R \xrightarrow{\Delta_R} R\times R \xrightarrow{\cdot} R$ and thus the polynomial $x^2+xy+y^2 \in \mathbb{Z}[X,Y]$ as the composite arrow

   $$R\times R \xrightarrow{\left\langle (-)^2\pi_1, (-)\cdot(-), (-)^2\pi_2 \right\rangle} R\times R\times R \xrightarrow{+} R$$

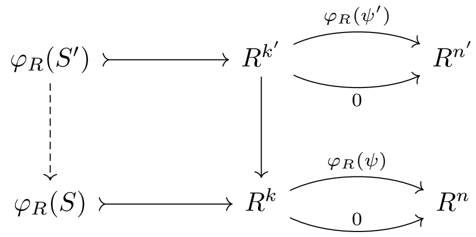
   Then we put together these n maps $R^k \to R$ to give a product map $R^k \to R^n$ which is the definition of φ_R on the original ring map.
4. Finally, every ring S in the category FinRing is isomorphic to one of the form $\mathbb{Z}[W_1,\ldots,W_k]/\langle P_1,\ldots,P_n\rangle$ which can be defined as the coequaliser

   $$\mathbb{Z}[X_1,\ldots,X_n] \underset{0}{\overset{\psi}{\rightrightarrows}} \mathbb{Z}[W_1,\ldots,W_k] \longrightarrow \mathbb{Z}[W_1,\ldots,W_k]/\langle P_1,\ldots,P_n\rangle$$

   where $\psi(X_i) = P_i$. Since φ_R is supposed to be left exact, it should preserve equalisers in $\mathrm{FinRing}^{\mathrm{op}}$ which are coequalisers in FinRing so $\varphi_R(S) := \varphi(\mathbb{Z}[W_1,\ldots,W_k]/\langle P_1,\ldots,P_n\rangle)$ is defined to be equaliser of the diagram



and this equaliser construction then easily allows φ_R to be defined on all maps between rings $S \to S'$ in FinRing (supposing $S' \cong \mathbb{Z}[W'_1, \ldots, W'_{k'}]/\langle P'_1, \ldots, P'_{n'}\rangle$)



where the map $R^{n'} \to R^n$ is the image under φ_R of the underlying map $\mathbb{Z}[W_1, \ldots, W_k] \to \mathbb{Z}[W'_1, \ldots, W'_{k'}]$ and the dotted map is the unique one that must exist by the universal property of equalisers.

$\varphi_R(\mathbb{Z}) := 1$ as it is the empty product of $\mathbb{Z}[X]$ and since limits commute with limits, our definition of φ_R on general rings by an equaliser must preserve binary products. We also know by construction that φ_R preserves certain equalisers and with more work can be shown to preserve all of them. Thus φ_R is seen to be left exact.

By construction we have $\mathrm{ev}_{\mathbb{Z}[X]}(\varphi_R) = R$ and it can be shown in steps similar to the construction of φ_R that any left exact functor $\mathrm{FinRing}^{\mathrm{op}} \to \mathcal{E}$ is naturally isomorphic to $\varphi_{F(\mathbb{Z}[X])}$, showing that indeed we do have an equivalence $\mathrm{Lex}(\mathrm{FinRing}^{\mathrm{op}}, \mathcal{E}) \cong \mathrm{Ring}(\mathcal{E})$ as desired.

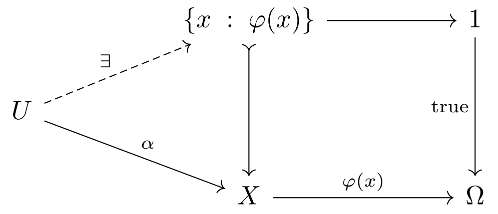
Example 6.3 (Local Rings)**.** Finally, the big Zariski topos, discussed in **Example 1.3**, can be shown by an application of **Pseudo-Diaconescu** and some work restricting the equivalence described above to be the classifying topos of the theory of local rings!

7 Internal Logic & Geometry

We end this essay with a brief further discussion of the internal language of a topos, previously mentioned in **Example 4.2**. The author is of the opinion that if any reader wonders what use Topos Theory or perhaps more generally Category Theory is to other, more traditional areas of mathematics, the answer is to comprehend the application of reasoning internally. In this section, we look at the case study of interpreting the statement of the Weierstrass Approximation Theorem internally to the topos $\mathrm{Sh}(T)$ of sheaves on a topological space T.

A proper explanation of everything involved in this section would require a lot more work than is feasible to explain in this essay so we will regularly be stating theorems from the literature, such as from [6] (which inspired the inclusion of this section). Indeed the central issue is that, as seen in **Example 4.2**, unpacking the "external" meaning in an elementary topos $\mathcal{E}$ of even a simple statement written in the internal language of $\mathcal{E}$ can be rather complicated. In order to make process of unraveling the meaning of internal statements simpler the notion of the **Kripke-Joyal semantics** was developed using the following definition.

Definition 7.1 (Forcing)**.** Let $\varphi(x) : X \to \Omega$ be a formula in the internal language of an elementary topos $\mathcal{E}$ with one free variable of type X and let $\alpha : U \to X$ be a term of type X. Then we define the statement $U \Vdash \varphi(\alpha)$, pronounced "U forces $\varphi(\alpha)$", to hold iff $\operatorname{im}\alpha \leq \{x : \varphi(x)\}$, i.e. iff α factors through $\{x : \varphi(x)\}$ as in



This definition can easily be extended as needed to formulas with more variables, e.g. for $\varphi(x, y)$ and terms $\alpha : U \to X, \beta : U \to Y$ we say that $U \Vdash \varphi(\alpha, \beta)$ iff the map $\langle \alpha, \beta \rangle : U \to X \times Y$ factors through $\{(x, y) : \varphi(x, y)\}$. On the other hand, note that if $\varphi : 1 \to \Omega$ is a formula with no free variables then $1 \Vdash \varphi$ iff $\varphi = \text{true}$, i.e. iff it is valid.

The name **forcing** comes from the fact that categorical logic can implement a version of Cohen's method of forcing used to prove the independence of the Continuum Hypothesis from ZFC in which this idea becomes relevant.

Now we in this section are primarily interested in the situation where our topos $\mathcal{E}$ is the category $\mathrm{Sh}(T)$ for a space T so that objects in $\mathcal{E}$ are sheaves on T. In this context, it turns out that the fact that all presheaves are colimits of representables means that for the most part the only generalised elements α we need to discuss are natural transformations from representable functors $\mathrm{Hom}(-, U) \implies X$ (for some open subset $U \subseteq T$) or equivalently, by Yoneda, elements $\alpha \in X(U)$. Using the notation α in this way both for a natural transformation $\mathrm{Hom}(-, U) \implies X$ and an element $\alpha \in X(U)$ tells us that $U \Vdash \varphi(\alpha) \iff \alpha \in \{x : \varphi(x)\}(U)$.

Note that in this context $\{x : \varphi(x)\}$ is a subsheaf of the sheaf X so $\alpha \in \{x : \varphi(x)\}(U)$ means that for an open subset $V \subseteq U$ we have that $\alpha|_V \in \{x : \varphi(x)\}(V)$. This is the **monotonicity** property of forcing: $U \Vdash \varphi(\alpha)$ implies, for $V \subseteq U$ open, $V \Vdash \varphi(\alpha|_V)$.

Similarly the uniqueness part of the definition sheaf tells you that if $\mathcal{U} = \{V_j\}_{j \in \mathcal{J}}$ is an open cover of U such that $V_j \Vdash \varphi(\alpha|_{V_j})$ for all $j \in \mathcal{J}$ then $U \Vdash \varphi(\alpha)$; this is the so-called **local character** of forcing.

Then the statement of Kripke-Joyal semantics for the special case of the category $\mathrm{Sh}(T)$ is:

Theorem 7.1 (*Kripke-Joyal*)

Let X be a sheaf on T, $\varphi_i(x)$ for $i \in \mathcal{I}$ be formulas in the language of the topos $\mathrm{Sh}(T)$ with x being a free variable of type X and $\alpha \in X(C)$. Then for $U \subseteq T$ an open subset:

- $U \Vdash \bigwedge_{i \in \mathcal{I}} \varphi_i(\alpha)$ iff $U \Vdash \varphi_i(\alpha)$ for all $i \in \mathcal{I}$.
- $U \Vdash \bigvee_{i \in \mathcal{I}} \varphi_i(\alpha)$ iff there is a open cover $\mathcal{U} = \{V_j\}_{j \in \mathcal{J}}$ of U such that for each $j \in \mathcal{J}$ we have $V_j \Vdash \varphi_i(\alpha)$ for some $i \in \mathcal{I}$.
- $U \Vdash \varphi_1(\alpha) \implies \varphi_2(\alpha)$ iff for all open subsets $V \subseteq U$, if $V \Vdash \varphi_1(\alpha|_V)$ then $V \Vdash \varphi_2(\alpha|_V)$.
- $U \Vdash \bot$ iff $U = \emptyset$ and thus $U \Vdash \neg\varphi_1(\alpha)$ iff the only open $V \subseteq U$ such that $V \Vdash \varphi_1(\alpha)$ is $V = \emptyset$.

  Regarding quantifiers, let $\varphi(x, y)$ be a formula with variables of types X, Y:
- $U \Vdash (\exists\, y \in Y)\varphi(\alpha, y)$ iff there is a covering $\{V_j\}_{j \in \mathcal{J}}$ of U and an element $\beta_j \in Y(V_j)$ such that $V_j \Vdash \varphi(\alpha, \beta_j)$ for each $j \in \mathcal{J}$.
- $U \Vdash (\forall\, y \in Y)\varphi(\alpha, y)$ iff for every open $V \subseteq U$ and element $\beta \in Y(V)$ we have $V \Vdash \varphi(\alpha, \beta)$.

This can be derived from Theorem 7.1 in [12] Chapter VI or is discussed explicitly in [3] §2.1. This result and the discussion preceding it can be summarised by the slogan that in topoi of sheaves on a space *truth and validity is local.* This gives us an incredibly helpful dictionary with which to unwrap internal statements more easily.

We are in particular interested in the **Weierstrass Approximation Theorem** from real analysis, which tells you that any continuous function $f : [0,1] \to \mathbb{R}$ can be approximated uniformly arbitrarily well by polynomials or, in symbols,

$$(\forall\, f \in C([0,1],\mathbb{R}))(\forall\, \varepsilon \in \mathbb{R}_{>0})(\exists\, P \in \mathbb{R}[X])(\forall\, t \in [0,1])\ \ |f(t) - p(t)| < \varepsilon$$

and remarkably it's possible to make sense of this internally in $\mathrm{Sh}(X)$.

The general idea for much of this is that we can simply repeat the construction of the objects in question that we do in traditional set theory by writing down the logical statements and formulas etc. in the internal language of $\mathrm{Sh}(T)$ and then working out what these constructions actually do in the category. Our first example we saw of this was given in **Example 4.2** where we showed that the internal statement for a morphism f suggesting that it is an injective function externally becomes the statement that it is a monomorphism.

Similarly to define the reals, we begin by considering the sheafification of the constant presheaf $\mathbb{N}$ to be the natural numbers internally. Indeed, this object can be singled out as satisfying the universal property of a **natural numbers object**, i.e. an object $\mathbf{N}$ in $\mathrm{Sh}(T)$ equipped with a morphism $1 \to \mathbf{N}$ and a morphism $s : \mathbf{N} \to \mathbf{N}$ such that for any other diagram $1 \xrightarrow{a} A \xrightarrow{f} A$ there is a unique morphism $u : \mathbf{N} \to A$ making the diagram

$$\begin{array}{ccccc} 1 & \longrightarrow & \mathbf{N} & \xrightarrow{\ s\ } & \mathbf{N} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\ 1 & \xrightarrow{\ a\ } & A & \xrightarrow{\ f\ } & A \end{array}$$

We can think of the arrow $1 \to A$ specifying an initial condition for the endomorphism $A \xrightarrow{f} A$, together defining a kind of dynamical system, in which case the natural numbers with $\mathbf{N} \xrightarrow{s} \mathbf{N}$ (which should be thought of as the successor map) here are considered to be the *universal dynamical system*. In the case of the constant sheaf $\mathbb{N}_T$, the arrows $1 \to \mathbb{N}_T, \mathbb{N}_T \to \mathbb{N}_T$ are the maps $\{1\} \mapsto 1 \in \mathbb{N}, (n \mapsto n+1)$ sent through the sheafification functor.

Using these natural numbers, we can define an internal version of the integers $\mathbb{Z}_T$ by the formula

$$\{(n,m)\ :\ n, m \in \mathbb{N}\}/\sim$$

where $\sim$ is the equivalence relation $(n,m) \sim (n',m')$ iff $n + m' = n' + m$. Encoding this in more detail exhibits the integers as a certain coequaliser $E \rightrightarrows \mathbb{N}_T \times \mathbb{N}_T \to \mathbb{Z}_T$ where E encodes the equivalence relation as the pullback

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{N}_T \times \mathbb{N}_T \\ \downarrow & & \downarrow{\scriptstyle +} \\ \mathbb{N}_T \times \mathbb{N}_T & \xrightarrow{\ +\ } & \mathbb{N}_T \end{array}$$

It can then be checked that the object this defines is the same as the constant sheaf $\mathbb{Z}$. Continuing in this way, we can write down the traditional construction of the rational numbers as equivalence classes of pairs (n,m) of integers and it turns out to be equivalent to the constant sheaf $\mathbb{Q}$ on T.

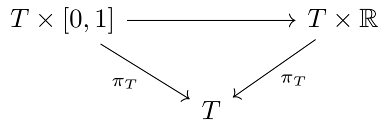
Finally, the real numbers are constructed internally by using precisely the normal axiomatisation of what it means for a pair (L,U) of subobjects $L, U \rightarrowtail \mathbb{Q}_T$ to form a Dedekind cut, i.e. there is some $q \in \mathbb{Q}$ in L and some $r \in \mathbb{Q}$ in U, L is downward closed and U upward closed etc. so that internally we have the (complicated) definition

$$\mathbb{R}_T := \{(L,U) \in P(\mathbb{Q}_T) \times P(\mathbb{Q}_T)\ :\ (L,U) \text{ is a Dedekind cut}\}$$

This is quite a remarkable idea. $\mathbb{R}_T$ is, externally, just a certain sheaf on the topological space T and thus to most mathematicians looks quite far from the reals. However, in the internal logic of $\mathrm{Sh}(T)$ it looks a lot like them! Internally a sheaf of sets looks like a set and in fact $\mathbb{R}_T$ can be shown to be a local ring object in $\mathrm{Sh}(X)$, i.e. an internal local ring, and it can be given an internal linear ordering (which externally is a certain "linear order" object $< \rightarrowtail \mathbb{R}_T \times \mathbb{R}_T$).

In fact, this construction of $\mathbb{R}_T$ ends up producing a rather familiar sheaf, as shown in [6], Theorem 8.8 and the subsequent discussion in Theorem 8.23:

Theorem 7.2

> In Sh(T), the real numbers object $\mathbb{R}_T$ is the sheaf $C(-, \mathbb{R})$ of continuous functions from subsets of T to $\mathbb{R}$. Furthermore, internal continuous functions $[0, 1] \to \mathbb{R}$ correspond bijectively with functions on the topmost edge of commuting triangles of the form
>
> 
>
>
> or equivalently as continuous functions $T \times [0, 1] \to \mathbb{R}$.

Now at last, we can try to explain what the statement of the Weierstrass Approximation Theorem might mean internally to Sh(T). Well, as discussed above, since it is a variable-free formula it is valid in this topos iff the terminal object forces it, i.e.

- $T \Vdash (\forall\, f \in C([0, 1], \mathbb{R}))(\forall\, \varepsilon \in \mathbb{R}_{>0})(\exists\, P \in \mathbb{R}[X])(\forall\, t \in [0, 1])\ \ |f(t) - P(t)| < \varepsilon$
- But then applying our $\forall$ rule given by Kripke-Joyal and **Theorem 7.2** this is equivalent to for all open subsets $U \subseteq T$ and continuous functions $f : U \times [0, 1] \to \mathbb{R}$,

$$U \Vdash (\forall\, \varepsilon \in \mathbb{R}_{>0})(\exists\, P \in \mathbb{R}[x])(\forall\, t \in [0, 1])\ \ |f(t) - P(t)| < \varepsilon$$

- Again applying Kripke-Joyal, we get the statement that for all open subsets $U \subseteq T$, continuous functions $f : U \times [0, 1] \to \mathbb{R}$ and for all open subsets $V \subseteq U$ and continuous functions $\varepsilon(-) : V \to \mathbb{R}_{>0}$ we have

$$V \Vdash (\exists\, P \in \mathbb{R}[X])(\forall\, t \in [0, 1])\ \ |f(t) - P(t)| < \varepsilon$$

- $\mathbb{R}[X]$ is internally encoded as the subobject of $\mathbb{R}^{\mathbb{R}}$ consisting of the polynomial functions $\mathbb{R} \to \mathbb{R}$, i.e. we use the internal ring operations on $\mathbb{R}_T$ to describe it as the union of the subobject of degree n polynomials $\{f : (\exists a_0 \in \mathbb{R})(\ldots)(\exists a_n \in \mathbb{R}) f = a_n X^n + \ldots + a_0\}$ for each n (as discussed in **Example 6.2**). Then we can use the $\exists$ rule of Kripke-Joyal to get that the above is equivalent to for all open subsets $U \subseteq T$, all continuous functions $f : U \times [0, 1] \to \mathbb{R}$, all open subsets $V \subseteq U$, all positive continuous functions $\epsilon(-) : V \to \mathbb{R}$ there is an open covering $\{V_j\}_{j \in \mathcal{J}}$ of V such that for each V_j there exists a function $P : U \times \mathbb{R} \to \mathbb{R}$, where $P_x(-) := P(x, -) : \mathbb{R} \to \mathbb{R}$ for each $x \in U$ is a polynomial function, where

$$V_j \Vdash (\forall t \in [0, 1])\ |f(t) - P(t)| < \varepsilon$$

- With the last quantifier unravelled, we get the statement that for all continuous $f : U \times [0, 1] \to \mathbb{R}$ (for $U \subseteq T$ open) and all positive functions $\varepsilon(-) : V \to \mathbb{R}_{>0}$ defined on an open subset $V \subseteq U$, there exists a function $P_j : V_j \times \mathbb{R} \to \mathbb{R}$ which restricts for each $x \in V_j$ to a polynomial $\mathbb{R} \to \mathbb{R}$ for each V_j in an open cover $\{V_j\}_{j \in \mathcal{J}}$ such that on any open subset $W \subseteq V_j$ and choice of real function $t : T \to \mathbb{R}$ we have

$$|f(x, t(x)) - P_j(x, t(x))| < \varepsilon$$

  for all $x \in W$.

This is admittedly a bit of a mouthful but can simplified by setting $V = U = T, W = V_j$, $\varepsilon : X \to \mathbb{R}$ to just be a positive constant function and $t : W \to \mathbb{R}$ also constant to get:

Theorem 7.3 (*Sheafy Weierstrass*)

> For any continuous function $f : T \times [0, 1] \to \mathbb{R}$ and any $\varepsilon > 0$, there is an open cover $\{V_j\}_{j \in \mathcal{J}}$ of T such that on each V_j there is a continuous function $P_j : V_j \times [0, 1] \to \mathbb{R}$, such that for each $x \in T$ we have $P_j(x, -) : V_j \times [0, 1] \to \mathbb{R}$ is a polynomial function,
>
> $$\|f(x, -) - P_j(x, -)\|_\infty < \varepsilon$$

In fact, this statement is true. As alluded to previously, any proof done in intuitionistic set theory is valid in the internal logic of any topos and [2] exhibits a constructive proof of in fact the more general Stone-Weierstrass Theorem precisely for using it in the internal language of topoi[2] so simply interpreting the steps in that proof internally in $\mathrm{Sh}(T)$ we get a proof of **Theorem 7.3**!

Notice that the traditional **Weierstrass Approximation Theorem** would tell you that each $f_x := f(x,-) : [0,1] \to \mathbb{R}$ can be approximated uniformly to within ε error with some polynomial but not that we can manage to simultaneously approximate the family (f_x) by another family of polynomials that can be put together into a continuous function on any open subset of T. It seems plausible that with some work the traditional proof using Bernstein polynomials[3] could be made to yield **Theorem 7.3** given the coefficients have a continuous dependence on the choice of function to approximate but the elegance of the internal method is that essentially no extra work is needed - as long as your original proof was intuitionistic you can reinterpret the proof in a topos of your choice and get all kinds of generalisations of your result!

8 Further Reading

This essay has been an introduction to the theory of topoi but has had to be brief on a number of matters and completely omit others. This final section aims to give some advice to the reader left yearning for more:

1. Topoi specifically of the form $\mathrm{Sh}(T)$ for some topological space T are particularly interesting in the category of Grothendieck topoi. Chapter IX of [12] discusses this and the more general case of T being a *locale*. This is the guiding perspective in [16], which acts as an introduction to Grothendieck toposes with more emphasis on the proof theory of geometric logic and spatial intuition.
2. Having learnt about classifying topoi, one may turn the line of questioning around and ask what various Grothendieck topoi of interest classify. [8] discusses this for various examples from algebraic geometry, for example the *crystalline topos* involved in defining crystalline cohomology.
3. For those interested in the use of internal logic in other areas of mathematics, the work of C.J. Mulvey has been entirely based on this often with an analytic flavour, e.g. internal results on Banach spaces. More recently, Ingo Blechschmidt wrote his PhD thesis [3] on simplifying ideas in basic scheme theory using the internal language of various topoi, culminating in some serious improvements in the complexity of proofs of results such as Grothendieck's generic freeness lemma. It has come to the author's attention that Chris Grossack [7] has also previously discussed the Weierstrass Approximation Theorem internally though attempts to employ a different proof technique involving *double-negation sheaves*, an idea from elementary topos theory beyond the scope of this essay.

[2] This is part of the second author's (Mulvey) program to develop the theory of Gelfand duality internally to topoi.

[3] Incidentally, this proof is not, at least without some meaningful change, intuitionistically valid as the error-analysis involved at one point splits various intervals in $[0,1]$ into "bad" or "good" and knowing that intervals are either bad or good requires applying the Law of the Excluded Middle.

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