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1 Introduction

"If life is going to exist in a Universe of this size, then the one thing it cannot afford to have is a sense of proportion." This is a quote taken from Douglas Adams' "The Hitchhiker's Guide to the Galaxy", where the main characters come face-to-face with the Total Perspective Vortex. It's simply a machine that shows you a map of the universe, including an innocuous, but tiny, "You are Here". Yet, it becomes used as a frighteningly efficient killing device - mere mortals are unable to grasp the totality of creation.

Luckily, humans have developed a way to get around this without our heads exploding - we're amazing at spotting patterns! They allow us to compress the world, reshape it into simple stories, by regarding things as the same (or related) even if they technically aren't - something which has been argued to be the very essence of mathematical thinking.

And one of the most obvious patterns is *symmetry*. There are various evolutionary reasons why we find it visually pleasing, but it took us until the 20th century to truly appreciate the deep role it has to play within physics, via the work of Emmy Noether and her eponymous theorem. Against the infinite abyss, symmetry plays the role of our saviour, tightly constraining the behaviour of physical systems and allowing us to describe the world efficiently.

But every hero needs a villain, and we find one in the strong force. Its quantum mechanics not only breaks our classical intuition, but classical symmetry too! The gauge coupling grows wild and untamed at the energy scales of our world, breaking classical scale-invariance. Nature conspires to hide quarks forever from our grasp through confinement. Far from the realm of spherical cows and Taylor series, what's a self-respecting physicist to do?

It turns out what we need is trust, and a little patience. By carefully working through the mathematics, we'll see that the broken pieces Nature leaves us contain remnants of the original theory. With playfulness and a little creativity, we'll morph, mold and twist these fragments into familiar constituents of matter - the proton, and the neutron. Ultimately, we'll come to appreciate that the *breaking* of symmetry has just an important role to play as symmetry itself, and find it fruitful to allow them to coexist as we continue attempting to understanding the world around us.

2 The Symmetries of QCD

2.1 Asymptotic Freedom

We introduce the quantum field theory that we'll be studying throughout this essay - QCD! Specifically, we consider a $SU(N_c)$ gauge theory coupled to N_f Dirac fermions, which we'll refer to as quarks, each in the fundamental representation of the gauge group. This has Lagrangian

$$\mathcal{L} = -\frac{1}{2g_s^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f}\bar{\psi}_i(i\not{D} - m_i)\psi_i \tag{2.1.1}$$

, where we've taken the strong coupling constant g_s to sit in the denominator, so that the covariant derivative takes the form

$$\not{D}\psi = \not{\partial}\psi - i\gamma^\mu A_\mu\psi \tag{2.1.2}$$

, and we've suppressed colour indices everywhere. The strong force takes the values $N_c = 3$ and $N_f = 6$, though we'll see later that it's convenient to treat N_f as 2 or sometimes 3 for our purposes.

Classically, we can set g_s to be small and study a theory of weakly-interacting quarks. But in the quantum theory the coupling runs, increasing with energy, so that the theory becomes strongly coupled at the energy scales relevant to our world. This phenomenon is called *asymptotic freedom*, and has to do with the beta function of QCD being negative for the physically relevant values of N_c and N_f.

We can associate a energy-scale Λ_{QCD} at which the coupling appears to diverge - around 200 MeV in our world. At lower energies, quarks must confine into mesons $q^a \bar{q}_a$ or baryons $\epsilon^{abc} q_a q_b q_c$, and we can no longer trust the results of perturbation theory. Instead, we must return to 2.1.1 and see how much symmetry allows us to constrain and control the physics.

2.2 The Chiral Limit

We'll work in the chiral limit of massless quarks. This allows us to split the Dirac fermions in 2.1.1 into pairs of non-interacting Weyl fermions:

$$\sum_{i=1}^{N_f} i\bar{\psi}_i \not{D}\psi_i = \sum_{i=1}^{N_f} i\bar{\psi}_{Ri}\bar{\sigma}^\mu \mathcal{D}_\mu \psi_{Ri} + i\bar{\psi}_{Li}\sigma^\mu \mathcal{D}_\mu \psi_{Li} \tag{2.2.1}$$

This classical Lagrangian thus has a global *flavour* symmetry, mixing the quark species,

$$G_F = U(N_f)_L \times U(N_f)_R \cong U(1)_V \times U(1)_A \times SU(N_f)_L \times SU(N_f)_R \tag{2.2.2}$$

, with a group action

$$U(N_f)_L : \psi_{Li} \to L_{ij}\psi_{Lj} \text{ and } U(N_f)_R : \psi_{Ri} \to R_{ij}\psi_{Rj}$$

Real-world QCD is not described by such a Lagrangian, but it's a convenient fiction to pretend that *some* quarks, particularly those lighter than Λ_{QCD}, are massless, and then treat their masses as a perturbation. In our world these are the up, down and (possibly) strange quarks, with masses 2 MeV, 5 MeV and 93 Mev respectively. So we'll consider $N_f = 2$ or sometimes $N_f = 3$.

We've also made use of the isomorphism $U(N) \cong U(1) \times SU(N)$, factoring out the determinant of the unitary matrix as an overall phase. It's convenient to arrange the $U(1)_L \times U(1)_R$ instead as the product $U(1)_V \times U(1)_A$, corresponding to the *vector* and *axial* symmetries respectively:

$$\begin{aligned} U(1)_V &: \psi_{L,i} \to e^{i\alpha}\psi_{L,i} \quad \psi_{R,i} \to e^{i\alpha}\psi_{R,i} \\ U(1)_A &: \psi_{L,i} \to e^{-i\beta}\psi_{L,i} \quad \psi_{R,i} \to e^{i\beta}\psi_{R,i} \end{aligned}$$

The conserved currents, written in terms of Dirac fermions, are $i\bar{\psi}\gamma^\mu\psi$ for $U(1)_V$, transforming as a Lorentz vector, and $i\bar{\psi}\gamma^\mu\gamma^5\psi$ for $U(1)_A$, transforming as an axial vector.

Why factor out the phases this way? It turns out that, while $U(N_f)_L \times U(N_f)_R$ is the correct global flavour symmetry group of the classical theory, it is *not* that of the quantum theory. The axial symmetry, like classical scale invariance, suffers an anomaly - the *chiral* anomaly.

2.3 The Chiral Anomaly

2.3.1 The Fermionic Path Integral

How might the quantum theory kill a classical symmetry? The key difference is the path integral - rather than focusing on stationary points of the action S, we sum over all field configurations weighted by $e^{iS/\hbar}$. So it must be that the "sum" $\mathcal{D}\phi$ is what explicitly breaks the symmetry, and we'll need to be careful in defining what it actually means (closely following the arguments of Fujikawa [1] and section 3 of [2]).

For fermions the measure is

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \tag{2.3.1}$$

as appropriate for complex Grassmann variables. Under a $U(1)_A$ transformation $\psi \to e^{i\alpha\gamma^5}\psi$, the effect is to change variables to $\psi' = \psi + i\alpha\gamma^5\psi$ and $\bar{\psi}' = \bar{\psi} + i\alpha\gamma^5\psi$, infinitesimally. It turns out the anomaly will arise from the *Jacobian* of this change of variables.

To be quantitative, we need to specify what (2.3.1) actually means. Here we get a surprising result - the definition of the measure actually depends on the action we use! In $d = 0 + 1$-dimensional quantum mechanics, for example, we have mathematically that "$\mathcal{D}\phi$" is ill-defined, but $\mathcal{D}\phi e^{iS[\phi]}$, after a suitable Wick-rotation, can actually be made rigorous as a Wiener measure, which appears in the construction of Brownian motion.

In fact, the way that Wick rotations for fermions work is a little different to what we might be used to for scalars, stemming from the fact that the Dirac action is *first*-order and not second-order. It's worth just laying out explicitly how the procedure works:

- We start out with our partition function $Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[\psi,\bar{\psi}]}$ where $S[\psi,\bar{\psi}] = \int d^4x\bar{\psi}(i\not{D} - m)\psi$, where we've added a mass term for generality. This also makes clear that the Dirac operator $i\not{D}$ is Hermitian in Minkowski space.

- As is standard, we define $\tau = it$, so $t = -i\tau$. Thus, the measure changes as $\int d^4x \to (-i)\int d^4x_E$, cancelling the i in the exponent from the path integral.

- Investigating the partial derivatives more carefully, we have an $i\not{\partial} = i\gamma^\mu\partial_\mu$ term. The spatial derivatives $\partial_i \to \partial_i$ are unaffected, but the temporal derivative goes as $\partial_0 \to i\partial_4$.

- To keep the form of the Dirac action, we absorb the $-i$ factor into the γ^0 matrix, defining $\gamma^4 = i\gamma^0$

- As for the gauge field(s), since it appears as $\gamma^\mu A_\mu$, we have $\gamma^0 A_0 \to i\gamma^4 A_0 := \gamma^4 A_4$ for $A_4 = -iA_0$. The mass term is unaffected.

- The Dirac operator in the Euclidean theory is given by $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$, and this is the object that is now Hermitian.

Thus, the Euclidean path integral we'll be considering is

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{\int d^4x_E\bar{\psi}(i\not{D}-m)\psi} \tag{2.3.2}$$

Our Euclidean gamma-matrices $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ all *anti*-Hermitian with

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = -2\delta^{\mu\nu} \tag{2.3.3}$$

In particular, the definition of γ^5 changes from

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \to \gamma^4\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4 \tag{2.3.4}$$

This means the relevant trace identity is

$$\mathrm{Tr}(\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d) = -4\epsilon^{abcd} \tag{2.3.5}$$

in Euclidean space.

Treating "$\mathcal{D}\phi e^{S[\phi]}$" as a single object, we define $\mathcal{D}\phi$ by diagonalising the Dirac operator $\not{D}$. Being Hermitian, we expect it to have an orthonormal basis of complex-number valued eigen-spinors ϕ_n satisfying

$$\not{D}\phi_n = \lambda_n\phi_n, \int d^4x\bar{\phi}_n\phi_m = \delta_{nm} \tag{2.3.6}$$

, with real eigenvalues λ_n (here, we've left colour indices implicit). Performing a fixed change of basis to these eigen-spinors, with the coefficients of our fields defined as

$$\psi(x) = \sum_n a_n\phi_n(x), \bar{\psi}_n = \sum_n \bar{b}_n\bar{\phi}_n(x) \tag{2.3.7}$$

(where a_n and b_n are Grassmann-valued), we can absorb a fixed Jacobian into the def of the path integral measure, and define the fermionic measure to be

$$\prod_n \int d\bar{b}_n da_n \tag{2.3.8}$$

The advantage of doing so is that we can consider the Jacobian as being the determinant of some matrix labelled by the indices of a_n and b_n.

2.3.2 An Anomalous Measure

The infinitesimal change is $\delta\psi = i\alpha(x)\gamma^5\psi$, which expanded into eigen-spinors reads

$$\sum_n \delta a_n\phi_n = i\alpha(x)\sum_m a_m\gamma^5\phi_n \tag{2.3.9}$$

If we multiply on the left by $\bar{\phi}_m$ and integrate, using our orthogonality relation, we get that

$$\delta a_n = X_{nm} a_m \text{ with } X_{nm} = i \int d^4x \alpha(x) \bar{\phi}_n \gamma^5 \phi_m \implies a'_n = (\delta_{nm} + X_{nm}) a_m \tag{2.3.10}$$

Now, since $\int d\theta := \frac{\partial}{\partial \theta}$ for Grassmann variables, the measure changes by the *inverse* of the corresponding factor to what we're used to for ordinary (bosonic) variables, meaning the Jacobian factor is

$$J = \det{}^{-1}(\delta_{nm} + X_{nm}) \tag{2.3.11}$$

Finally, because we're considering the *axial* symmetry, we get the *same* Jacobian for the b_n transformation, meaning overall the measure changes as

$$\prod_n \int d\bar{b}_n da_n \to \prod_n \int d\bar{b}'_n da'_n = \prod_n \int d\bar{b}_n da_n J^2 \tag{2.3.12}$$

This also shows that $U(1)_V$ *is* a symmetry of the measure when the $U(1)_A$ symmetry is not - for the vector symmetry, the two components of the measure transform in opposite ways! Schematically, we'd have Jacobian factors $\det^{-1}(1+Y)$ and $\det^{-1}(1-Y)$ for ψ and $\bar{\psi}$ respectively, which vanish to leading order in α - the Lie Algebra-Lie Group correspondence via the exponential map will thus ensure that there's no overall factor for $U(1)_V$. We've found our first honest-to-goodness symmetry of the quantum theory, which goes by the catchier name of *baryon number*.

Making use of that correspondence allows us to actually compute the Jacobian for the axial symmetry transformation:

$$J = \det[(1+X)^{-1}] \approx \det(1-X) \approx \det(e^{-X}) = e^{-\mathrm{Tr}(X)} \tag{2.3.13}$$

, using Maclaurin series expansions inside the det together with $\det(e^{-X}) = \prod_\lambda e^{-\lambda} = e^{-\sum_\lambda \lambda} = e^{-\mathrm{Tr}(X)}$. The "trace" here is over spinor indices, colour indices and an overall integration over space, leaving us with a final expression

$$J = \exp(-i \int d^4x \alpha(x) \sum_n \bar{\phi}_n(x) \gamma^5 \phi_n(x)) \tag{2.3.14}$$

Worryingly, this is an infinite sum that could diverge, giving $J = \exp(-i\infty)$ which is obviously nonsense. But we're also taking a trace over spinor indices that should cause the argument of the exponential to vanish, since $\mathrm{Tr}(\gamma^5) = 0$, which naively suggests $J = \exp(0) = 1$. The correct way to reconcile this is, as always, finding a way to regulate the unphysical infinity - we'll do this by introducing a mass scale Λ.

We've kept things gauge-invariant throughout, and would like to preserve this as much as possible with the regulator Λ - thus, we'll couple it to the eigenvalue λ_n of each mode, and write

$$\begin{aligned}\int d^4x\alpha(x)\sum_n \bar{\phi}_n\gamma^5\phi_n &= \lim_{\Lambda\to\infty}\int d^4x\alpha(x)\sum_n \bar{\phi}_n\gamma^5\phi_n f(\lambda_n^2/\Lambda^2) \\ &= \lim_{\Lambda\to\infty}\int d^4x\alpha(x)\sum_n \bar{\phi}_n\gamma^5 f(\not{D}^2/\Lambda^2)\phi_n\end{aligned} \tag{2.3.15}$$

where $f(x)$ is some smooth function rapidly approaching 0 at $x = \infty$, with normalization $f(0) = 1$, and we've used the fact that ϕ_n is an eigenspinor of the Dirac operator $\not{D}$ with eigenvalue λ_n. An example of a suitable $f(x)$ is $f(x) = e^{-x}$.

Just before we dive into the algebra - the $\int d^4x$ term scales as Λ^4, so ideally we'll want to show that any $\frac{1}{\Lambda^0}$ and $\frac{1}{\Lambda^2}$ terms that come from expanding f vanish, leaving us with only the $\frac{1}{\Lambda^4}$ term, which integrated over space gives us a Λ^0 regulator-invariant contribution - this will be the anomaly.

At each point x the eigen-spinors ϕ_n form a basis, and we may write

$$\sum_n \bar{\phi}_n(x)\gamma^5 f(\not{D}^2/\Lambda^2)\phi_n(x) = \mathrm{Tr}_x\left(\gamma^5 f(\not{D}^2/\Lambda^2)\right) \tag{2.3.16}$$

This allows us to transform into a basis more suited for differentiation, namely that of plane waves. With the usual normalization for momentum integrals we can thus write

$$\mathrm{Tr}_x\left(\gamma^5 f(\not{D}^2/\Lambda^2)\right) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(e^{-ik\cdot x}\gamma^5 f(\not{D}^2/\Lambda^2)e^{ik\cdot x}\right) \tag{2.3.17}$$

where we still need to take a trace over spinor and colour indices. At this point, we need some identities that let us know how the covariant derivative acts on our plane waves:

$$\begin{aligned}\not{D}^2 = \gamma^\mu\gamma^\nu\mathcal{D}_\mu\mathcal{D}_\nu &= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\mathcal{D}_\mu\mathcal{D}_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\mathcal{D}_\mu\mathcal{D}_\nu \\ &= \eta^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu + \frac{1}{4}[\gamma^\mu,\gamma^\nu][\mathcal{D}_\mu,\mathcal{D}_\nu] \\ &= \mathcal{D}^2 - \frac{ie}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} \\ &= \mathcal{D}^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\end{aligned} \tag{2.3.18}$$

where we've used the Clifford algebra definition, that $[\gamma^\mu,\gamma^\nu]$ is antisymmetric in μ and ν, and the definition of the field strength tensor.

We also need

$$e^{-ik\cdot x}\mathcal{D}_\mu e^{ik\cdot x} = \mathcal{D}_\mu + ik_\mu \tag{2.3.19}$$

which can be derived by thinking of $e^{ik\cdot x}$ as a gauge transformation.

Combining these identities, we can write (2.3.17) as

$$\begin{aligned}&\lim_{\Lambda\to\infty}\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\gamma^5 f((\mathcal{D}_\mu+ik_\mu)^2/\Lambda^2-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2)\\=&\lim_{\Lambda\to\infty}\Lambda^4\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}\gamma^5 f((ik_\mu+\mathcal{D}_\mu/\Lambda)^2-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2)\end{aligned}\tag{2.3.20}$$

where we've substituted $k\to\Lambda k$ in the integral.

We're going to taylor-expand our regulator f around $(ik_\mu)^2=-k_\mu k^\mu$. Here, the trace over spinor indices will help - $\operatorname{tr}(\gamma^5)=0$ and $\operatorname{tr}(\gamma^5\gamma^\mu\gamma^\nu)=0$, so the first nonzero term in the expansion comes from squaring the $\gamma^\mu\gamma^\nu F_{\mu\nu}/\Lambda^2$. Note that this also comes with a $1/\Lambda^4$ factor that will give a finite contribution in the limit $\Lambda\to\infty$. Moreover, while we generically get other terms through the taylor expansion, these will be suppressed by higher powers of Λ and thus vanish in the limit.

So, we end up with

$$\begin{aligned}\sum_n\bar\phi_n\gamma^5\phi_n&=\operatorname{tr}\gamma^5\frac{1}{2!}(-\frac{ie}{2}\gamma^\mu\gamma^\nu F_{\mu\nu})^2\int\frac{d^4k}{(2\pi)^4}f''(-k_\mu k^\mu)\\&=\frac{e^2}{2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}_R(F_{\mu\nu}F_{\rho\sigma})\int\frac{d^4k}{(2\pi)^4}f''(-k_\mu k^\mu)\end{aligned}\tag{2.3.21}$$

where we've used $\operatorname{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)=-4\epsilon^{\mu\nu\rho\sigma}$ in Euclidean space with our conventions, and tr_R denotes a trace over colour indices.

It remains to do the k integral. Letting $x=-k_\mu k^\mu\geq 0$ in our negative definite Euclidean metric convention, we change to sphericals (picking up a factor of $\frac{2\pi^{4/2}}{\Gamma(4/2)}=2\pi^2$ from the surface area of S^3) and get

$$\int\frac{d^4k}{(2\pi)^4}f''(-k_\mu k^\mu)=\frac{1}{16\pi^2}\int_0^\infty dx x f''(x)\tag{2.3.22}$$

where we remember that $x=k^2$ to convert $dkk^3=\frac{1}{2}d(k^2)k^2=\frac{1}{2}dxx$,

$$=\frac{1}{16\pi^2}xf'(x)|_0^\infty-\frac{1}{16\pi^2}\int_0^\infty f'(x)dx=\frac{1}{16\pi^2}\tag{2.3.23}$$

where we've assumed that our regulator f satisfies

$$f(0)=1,f(\infty)=0,xf'(x)|_{x=0}=xf'(x)|_{x=\infty}=0$$

Substituting into 2.3.21 then gives

$$\sum_n\bar\phi_n\gamma^5\phi_n=\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}_R(F_{\mu\nu}F_{\rho\sigma})\tag{2.3.24}$$

Putting this all together, remembering we get *two* factors of the Jacobian, we have that the measure transforms as

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \to \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-\frac{ie^2}{16\pi^2}\int d^4x\alpha(x)\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_R(F_{\mu\nu}F_{\rho\sigma})\right) \tag{2.3.25}$$

This has the form of a so-called *mixed anomaly* between $U(1)_A$ and the gauge group for A_μ (which we've left unspecified).

2.3.3 The Anomalous Ward Identity

We can use how the measure changes to determine the new equation for the axial current. First, we check how the action changes under the change $\psi \to \psi' = \psi + \alpha X(\psi)$ for $X(\psi) = i\gamma^5\psi$, and $\bar{\psi} \to \bar{\psi}' = \bar{\psi} + \alpha X(\bar{\psi})$ for $X(\bar{\psi}) = i\bar{\psi}\gamma^5$ infinitesimally:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu(\alpha X(\psi)) + \frac{\partial\mathcal{L}}{\partial\psi}\alpha X(\psi) + \partial_\mu(\alpha X(\bar{\psi}))\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} + \alpha X(\bar{\psi})\frac{\partial\mathcal{L}}{\partial\bar{\psi}} \tag{2.3.26}$$

$\mathcal{L}$ only depends on $\partial\psi$ and $\bar{\psi}$ so two of these terms vanish, leaving us with:

$$\delta\mathcal{L} = (\partial_\mu\alpha)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}X(\psi) + \alpha[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu X(\psi) + X(\bar{\psi})\frac{\partial\mathcal{L}}{\partial\bar{\psi}}] \tag{2.3.27}$$

Since $\delta\mathcal{L} = 0$ when $\alpha = const$, the term proportional to α vanishes. Thus, the action $S = \int d^4\mathcal{L}$ changes as

$$\delta S = \int d^4x\delta\mathcal{L} = \int d^4x(\partial_\mu\alpha)J^\mu = -\int d^4x\alpha\partial_\mu J^\mu \tag{2.3.28}$$

with $J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}X(\psi) = -\bar{\psi}\gamma^\mu\gamma^5\psi$.

The partition function then changes as follows:

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{S[\psi,\bar{\psi}]} \to \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-\frac{ie^2}{16\pi^2}\int d^4x\alpha(x)\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_R(F_{\mu\nu}F_{\rho\sigma})\right)$$
$$e^{S[\psi,\bar{\psi}]}\exp\left(\int d^4x\alpha(x)\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)\right) \tag{2.3.29}$$

But since our theory is well-defined, the partition function is invariant under any field redefinitions, so the exponential factor must be 1. Taylor-expanding it and taking the linear term means that we must have:

$$\left\langle\int d^4x\alpha(x)\left(-\frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_R(F_{\mu\nu}F_{\rho\sigma}) + \partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)\right)\right\rangle = 0 \tag{2.3.30}$$

where we've carried out the path integral to convert this to a correlation function.

Since this holds for all $\alpha(x)$, we then have the anomalous Ward identity

$$\partial_\mu \langle \bar{\psi}\gamma^\mu\gamma^5\psi\rangle = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_R(F_{\mu\nu}F_{\rho\sigma}) \tag{2.3.31}$$

in our convention for the Euclidean path integral. We see explicitly that the axial current is not conserved in the quantum theory.

2.3.4 The Symmetry Group of QCD

We've shown that $U(1)_A$ is not a global symmetry when a Dirac fermion is coupled to a background gauge field, whereas $U(1)_V$ is - we identified the latter with baryon number.

What of the nonabelian flavour symmetries $SU(N_f)_L \times SU(N_f)_R$? It turns out that there's another way to see the anomaly, in perturbation theory, through the use of so-called *triangle diagrams* - Feynman diagrams with the anomalous current on one leg and gauge currents on the others.

We won't go through the details, but the upshot is that there's no analogue of a "mixed anomaly" between a nonabelian global symmetry and a nonabelian gauge symmetry, since any triangle diagram necessarily contains either 1 global generator or one gauge generator, with the corresponding trace killing the contribution. Thus, our global symmetry group for the quantum theory is

$$G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R \tag{2.3.32}$$

3 The Spectrum of QCD

It seems that we've done an awful lot of work merely to verify that $U(1)_A$ is not a global symmetry of our theory, while $U(1)_V \times SU(N_f)_L \times SU(N_f)_R$ is. But we've learned an important lesson - the measure $\mathcal{D}\phi$ is not to be taken lightly, and looking into its properties provides useful results. In particular, it'll help us to prove some nontrivial properties of the spectrum of QCD.

3.1 A Positive-Definite Measure

First, we're going to shift our conventions slightly by making our γ matrices Hermitian for the Euclidean path integral. We take our previous expression and insert appropriate factors of i, so that the Euclidean path integral takes the form

$$Z = \int \mathcal{D}A \prod_{i=1}^{N_f} \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i e^{-S_{\mathrm{YM}} + \sum \bar{\psi}_i(\not{D}+m)\psi_i} \tag{3.1.1}$$

where S_{YM} is the $SU(N_c)$ Yang-Mills action. We'll be closely following section 5.6 of [2].

We've also given a common bare mass $m > 0$ to each quark - we could generalise further and assign the quarks different masses (as in real-world QCD), at the expense of some nastier algebra. Instead we'll focus more on the big-picture results that can be derived from thinking carefully about the fermionic measure.

Specifically, suppose we want to find a correlation function $\langle \mathcal{O}(x) \dots \mathcal{O}(y) \rangle$, of gauge-invariant operators $\mathcal{O}(x)$. Supposing for now that they don't depend on the fermions, we can perform the path integral over the fermions and derive

$$\langle \mathcal{O}(x) \dots \mathcal{O}(y) \rangle = \frac{1}{Z} \int \mathcal{D}A e^{-S_{\text{YM}}} [\det(\not{D} + m)]^{N_f} \mathcal{O}(x) \dots \mathcal{O}(y) \tag{3.1.2}$$

where the determinant arises from the analog of the Gaussian integral for Grassmann variables. If any $\mathcal{O}$ do depend on fermions then we replace fermion bilinears with propagators via Wick's theorem.

We can think of the fermions as changing the measure of the path integral over the gauge field, to

$$d\mu = \frac{1}{Z} \int \mathcal{D}A e^{-S_{\text{YM}}} [\det(\not{D} + m)]^{N_f} \tag{3.1.3}$$

An innocuous but surprisingly powerful observation is that this measure is positive-definite. The Yang-Mills kinetic term certainly satisfies this (since Yang-Mills free theory is stable after all), but the determinant term is a little more subtle. It turns out that eigenvalues of the Dirac operator come in pairs - whenever we have $i\not{D}\psi = \lambda\psi$, using $\{\gamma^5, \not{D}\} = 0$ we automatically get

$$i\not{D}(\gamma^5\psi) = -i\gamma^5\not{D}\psi = -\lambda\gamma^5\psi$$

If $\lambda \neq 0$ then these represent genuinely distinct eigenvectors of $i\not{D}$. The remaining vectors fall into n "zero modes", allowing us to write the determinant as

$$\det(\not{D} + m) = m^n \prod_\lambda (m - i\lambda)(m + i\lambda) = m^n \prod_\lambda (m^2 + \lambda^2) \tag{3.1.4}$$

which is positive-definite (for $m > 0$).

What's so important about this? It turns out this allows us to deduce nontrivial properties of the spectrum of QCD, which strong coupling normally forbids without numerics, experiment or some stronger form of symmetry. In particular, among the many gauge singlets (mesons and baryons) that confinement forces the quarks into, one stands out as the lightest - a pseudoscalar meson called the *pion*.

3.2 The Pion is the Lightest Particle

The pion is a pseudoscalar meson field given by

$$\pi = \bar{\psi}_i \gamma^5 \psi_j \tag{3.2.1}$$

picking some $i \neq j$. In QCD "the pion" really refers to 3 different particles - the charged $\pi^{\pm}$ and the neutral π^0, each with their own interesting forms of decay - but for our purposes we'll allow any i and j to refer to a "pion".

We'll split this proof into two parts; showing that the pion is lighter than the mesons, and showing that the pion is lighter than the baryons. In order to get the masses of the particles into the game, we use the following fact - at large distances, the corresponding two-point correlation function for the particle, in position space, should be dominated by the exchange of that particle (or strictly speaking by the lightest particle carrying the same quantum numbers), with the typical Yukawa-potential type scaling

$$\langle J^a_\mu(x)(J^b_\nu)^\dagger(y)\rangle \sim e^{-M|x-y|} \tag{3.2.2}$$

. Here $J^a_\mu = \bar{\psi}_i\gamma^\mu(T^a)^{ij}\psi_j$ schematically, though we'll also consider scalar and pseudoscalar particles with no Lorentz index μ.

First, we'll calculate the pion propagator,

$$\langle \pi(x)\pi^\dagger(y)\rangle = \int d\mu \mathrm{tr}[S(x,y)\gamma^5 S(y,x)\gamma^5] \tag{3.2.3}$$

tracing over both spinor and colour indices, where

$$S(x,y) = \langle x|\frac{1}{\not{D}+m}|y\rangle \tag{3.2.4}$$

is the fermion propagator (note that the propagators have come from contracting ψ with $\bar{\psi}$ a la Wick's theorem).

$(\gamma^5)^2 = 1$ so we can view $\gamma^5 S(y,x)\gamma^5$ as a similarity transformation. In fact, since $\{\gamma^5, \not{D}\} = 0$ it takes the form

$$\gamma^5 S(y,x)\gamma^5 = \gamma^5\langle y|(\not{D}+m)^{-1}|x\rangle\gamma^5 = \langle y|(-\not{D}+m)^{-1}|x\rangle = \langle x(\not{D}+m)^{-1}|y\rangle^\dagger$$

using that $\not{D}$ is anti-Hermitian, and that conjugation swaps the order of bras and kets. But we recognise this as $S(x,y)^\dagger$, meaning that the pion propagator takes the form

$$\langle \pi(x)\pi^\dagger(y)\rangle = \int d\mu \sum_{\text{color,spin}} |S(x,y)|^2 \geq 0 \tag{3.2.5}$$

The propagator is positive-definite, and we might already see hints that the fermion propagator contributes "maximally" to the sum, pushing down the mass of the pion.

Indeed, suppose we investigate the corresponding expression for a scalar meson $\sigma = \bar{\psi}_i\psi_j$, for $i \neq j$ the same indices as those carried by the pion. We now get

$$\langle \sigma(x)\sigma^\dagger(y)\rangle = \int d\mu \mathrm{tr}[S(x,y)S(y,x)] = \int d\mu \mathrm{tr}[S(x,y)\gamma^5 S(x,y)^\dagger\gamma^5] \tag{3.2.6}$$

We see that there's a sum over $|S(x,y)|^2$ as before, except the presence of the γ^5 matrices lead to various plus and minus signs as we trace over spinor indices. An application of the triangle inequality necessitates

$$\langle\sigma(x)\sigma^\dagger(y)\rangle \le \langle\pi(x)\pi^\dagger(y)\rangle \tag{3.2.7}$$

whence, from the form of the two-point function at large distances, gives

$$e^{-m_\sigma|x-y|} \le e^{-m_\pi|x-y|} \implies m_\sigma \ge m_\pi \tag{3.2.8}$$

(where we've taken only the leading-order asymptotic behaviour and neglected prefactors).

How about for baryons? With $N_c = 3$ baryons take the form

$$B = \epsilon^{abc}\psi^i_a\psi^j_b\psi^k_c \tag{3.2.9}$$

leaving the contraction of spinor indices implicit, with a, b, c colour indices and i, j, k flavour indices. The corresponding two-point correlation function will take the form

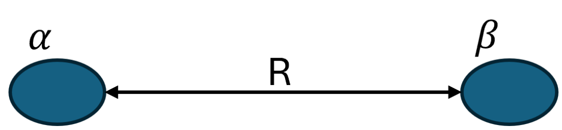
$$\langle B(x)B(y)^\dagger\rangle \sim e^{-m_B|x-y|} \tag{3.2.10}$$

asymptotically, with m_B the mass of the lightest baryon sharing the quantum numbers of B. The corresponding expression for the propagator is

$$\langle B(x)B(y)^\dagger\rangle = \epsilon^{abc}\epsilon^{a'b'c'}\int d\mu \mathrm{tr}[S(x,y)_{aa'}S(x,y)_{bb'}S(x,y)_{cc'}] \tag{3.2.11}$$

keeping colour indices explicit. We have an extra propagator factor here that we'd like to remove (up to prefactors) before comparing with the form of the pion propagator. To do this, it'll be useful to quickly prove a result that establishes a *uniform* bound on the propagator, independent of the gauge field, so that we can pull it out of $\int d\mu$.

To do this, we'll temporarily switch to talking about *smeared* wavepackets $|\alpha\rangle, |\beta\rangle$



of disjoint localised support, so that the gauge field $A_\mu(x)|\alpha\rangle = 0$ for x outside α, similarly for β.

We then rewrite the smeared propagator as

$$S(\alpha,\beta) = \langle\alpha|\frac{1}{\not{D}+m}|\beta\rangle = \int_0^\infty dt\langle\alpha|e^{-(\not{D}+m)t}|\beta\rangle = \int_0^\infty dt e^{-mt}\langle\alpha|e^{-i(-i\not{D})t}|\beta\rangle \tag{3.2.12}$$

where t is just an artificial dummy variable. The labelling is suggestive however - we can interpret t as a timelike direction for a theory in $d = 4+1$ dimensions with corresponding Hamiltonian $H = -i\not{D}$. Causality says that any signal from region α takes at least time $t = R$ (in natural units) to reach region β, meaning

$$\langle\alpha|e^{-iHt}|\beta\rangle = 0 \text{ for } 0 \leq t \leq R \tag{3.2.13}$$

At later 'times', we can use the Cauchy-Schwarz inequality to bound

$$|\langle\alpha|e^{-iHt}|\beta\rangle| \leq \sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta e^{+iHt}|e^{-iHt}\beta\rangle} = \sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle} \tag{3.2.14}$$

using unitarity of the time-evolution e^{-iHt} in the final step. This allows us to give a genuine uniform bound on the propagator

$$|S(\alpha,\beta)| \leq |\alpha||\beta| \int_R^\infty dte^{-mt} = \frac{|\alpha||\beta|}{m}e^{-mR} \tag{3.2.15}$$

We see that the bound explicitly involves $|\alpha||\beta|$ which disallows the use of non-normalisable position eigenstates and requires these smeared states - still, what ultimately matters is the asymptotic e^{-mR} dependence as $R \to \infty$.

Applying this bound for one of the S factors in our previous expression and neglecting prefactors, we get that

$$\langle B(\alpha)B^\dagger(\beta)\rangle \leq_\sim e^{-m|\beta-\alpha|} \int d\mu \sum_{\text{colour, spinor}} |S(x,y)|^2 = e^{-m|\beta-\alpha|}\langle\pi(\alpha)\pi(\beta)^\dagger\rangle \tag{3.2.16}$$

where we've abused notation to replace R by $|\beta - \alpha|$. Taking the disjoint regions further and further away and comparing the behaviours of the two-point functions, we learn that $m_B \leq m_\pi + m$, whence taking the bare mass as small, $m \to 0$ (say, compared to a natural lengthscale Λ_{QCD}) gives us the result

$$m_B \geq m_\pi \tag{3.2.17}$$

We've learned that, for some reason, the pseudoscalar meson of QCD is the lightest particle in the spectrum. But the proofs, while relatively short, are rather technical, and perhaps a little unenlightening. Ideally, there'd be a more concrete physical reason why the combination $\bar{\psi}\gamma^5\psi$ gives rise to such a light particle.

Happily, it turns out there is - but to find it, we'll need to look at our last encounter with the anomaly.

3.3 The Pion is a pseudo-Goldstone Boson

We saw previously that $U(1)_A$ is not a symmetry of QCD because it suffers an anomaly, from the coupling of the quarks to a $SU(3)$ gauge field, which can be viewed as a *mixed* $U(1) \times SU(3)^2$ anomaly. In contrast the vector $U(1)_V$ symmetry suffers no such anomaly with the gauge field - moreover, the $SU(N_f)_L \times SU(N_f)_R$ group is nonabelian, so cannot suffer a mixed anomaly with a nonabelian gauge group.

However, we can still compute anomalies for such global symmetry groups G_F, and these can be used to help link the UV theory with its (potentially complicated) IR theory, through what's known as t' Hooft anomaly matching. The argument can be found in the Standard Model lectures, but the takeaway for us is that either the anomaly of the IR theory must equal that of the UV theory, or the global symmetry group G_F gets spontaneously broken in the IR!

Why is this important? We assume the theory becomes strongly interacting in the IR, so that the only finite-energy states are mesons and baryons, with weak interactions betwween these. It turns out we'll able to show that, for $N_f = 3$ massless fermions, which we can approximate the up, down and strange quarks to be (at least, compared to Λ_{QCD}), it's impossible to reproduce the anomaly in the IR with massless baryons (mesons are bosons so don't contribute), which forces spontaneous symmetry breaking. This will provide an explanation for why pions are so light in QCD - they're precisely the (approximate) Goldstone bosons produced from this!

We'll continue following section 5.6 of [2]. Let's start by computing the UV anomalies - we've derived the chiral anomaly in detail, but the factors for other anomalies can be found in e.g. the Standard Model course.

Being a vector-like theory, QCD suffers no $[U(1)_V^3]$ anomaly, but we do get anomalies for the chiral factors. The first is a purely non-Abelian anomaly

$$[SU(N_f)_L]^3 : \mathcal{A} = \sum A(\square) = N_c \tag{3.3.1}$$

from the left-handed quarks transforming in the fundamental $\square$ of the flavour symmetry group, together with summing over the N_c colour indices. The right-handed quarks get the same anomaly.

We also get a mixed anomaly

$$[SU(N_f)_L]^2 \times U(1)_V : \mathcal{A}' = \sum qI(\square) = N_c \tag{3.3.2}$$

using the Dynkin index of the fundamental $I(\square) = 1$, and normalising the charges of the quarks under $U(1)_V$ to be 1. This again just gives the number of quark colours.

Now, we turn to the putative massless baryons that need to reproduce such an anomaly. We can take each constituent quark to be left-handed or right-handed, transforming as $(\mathbf{N}_f, \mathbf{1})$ and $(\mathbf{1}, \mathbf{N}_f)$ respectively under $SU(N_f)_L \times SU(N_f)_R \subset G_F$. We will tensor together 3 of these and decompose into irreps.

We'll need various group-theoretic factors for this:

- The dimension of a representation $\dim(\mathbf{R})$.
- The Dynkin index of a representation $\text{Tr}(T_R^A T_R^B) = \frac{1}{2} I(\mathbf{R}) \delta^{AB}$, normalised so that $I(\square) = 1$ for the fundamental.
- The anomaly coefficient of a representation,

  $$\text{Tr}(T_R^A \{T_R^B, T_R^C\}) = \mathcal{A}(\mathbf{R}) \text{Tr}(T^A \{T^B, T^C\})$$

  , where the latter denotes generators of the fundamental representation. Normalised so that $\mathcal{A}(\square) = 1$.

All of these have useful identities to do with taking conjugates, direct sums and tensor products of representations:

- $\dim(\overline{\mathbf{R}}) = \dim(\mathbf{R})$, $\dim(\mathbf{R}_1 \oplus \mathbf{R}_2) = \dim(\mathbf{R}_1) + \dim(\mathbf{R}_2)$, $\dim(\mathbf{R}_1 \otimes \mathbf{R}_2) = \dim(\mathbf{R}_1)\dim(\mathbf{R}_2)$
- $I(\overline{\mathbf{R}}) = I(\mathbf{R})$, $I(\mathbf{R}_1 \oplus \mathbf{R}_2) = I(\mathbf{R}_1) + I(\mathbf{R}_2)$, $I(\mathbf{R}_1 \otimes \mathbf{R}_2) = \dim(\mathbf{R}_1) I(\mathbf{R}_2) + \dim(\mathbf{R}_2) I(\mathbf{R}_1)$
- $\mathcal{A}(\overline{\mathbf{R}}) = -\mathcal{A}(\mathbf{R})$, $\mathcal{A}(\mathbf{R}_1 \oplus \mathbf{R}_2) = \mathcal{A}(\mathbf{R}_1) + \mathcal{A}(\mathbf{R}_2)$,
$$\mathcal{A}(\mathbf{R}_1 \otimes \mathbf{R}_2) = \dim(\mathbf{R}_1)\mathcal{A}(\mathbf{R}_2) + \dim(\mathbf{R}_2)\mathcal{A}(\mathbf{R}_1)$$

The conjugate identities for I and $\mathcal{A}$ may be proved by writing $\bar{T}^A_R = -T^{A\star}_R = -(T^A_R)^T$. By noting that generators of nonabelian gauge groups are always traceless, the direct sum and tensor product identities may be proved by writing $T^A_{R_1 \oplus R_2} = T^A_{R_1} + T^A_{R_2}$ and $T^A_{R_1 \otimes R_2} = T^A_{R_1} \otimes I + I \otimes T^A_{R_2}$, expanding out and collecting only the nonzero terms.

We need to worry about tensor products of up to 3 copies of the fundamental $\mathbf{3}$, so we need the identities

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} &= \overline{\mathbf{3}} \oplus \mathbf{6} \\ \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} &= \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{10} \end{aligned} \tag{3.3.3}$$

Using the identities above, together with $\mathcal{A}(\mathbf{3}) = I(\mathbf{3}) = 1$, we may derive the factors for each representation (listed in increasing order of quark number):

Representation	**Dimension**	**Dynkin Index**	**Anomaly Coefficient**
$\mathbf{3}$	3	1	1
$\overline{\mathbf{3}}$	3	1	-1
$\mathbf{6}$	6	5	7
$\mathbf{1}$	1	0	0
$\mathbf{8}$	8	6	0
$\mathbf{10}$	10	15	27

We may disregard the singlet from here on since it cannot contribute to the full nonabelian anomaly nor the mixed anomaly.

Next, we focus on handedness, and here we encounter our first subtlety. In massive QCD, baryons may have either spin $\frac{1}{2}$ or spin $\frac{3}{2}$ depending on how Lorentz indices get contracted. The analog for massless particles is helicity, and it turns out we need only worry about helicity $\pm\frac{1}{2}$ particles (i.e. left- or right-handed) due to a theorem by Weinberg and Witten [3]. This states that massless particles with helicity $> \frac{1}{2}$ cannot be charged under a gauge-invariant, Lorentz-covariant current - for our purposes, they must be singlets under the $U(1)_V$ flavour symmetry, and so cannot be baryons.

Thus, we'll need to contract the spin indices of two fermions of the same handedness, leaving the third spin degree of freedom hanging. For example, we can take three left-handed quarks to produce a left-handed bound state, or a left and two right-handed quarks (contracting the right-handed quarks), as well as the corresponding parity-reversed combinations.

It turns out we can list out the relevant left-handed baryons, together with their right-handed counterparts, as: 1. $\mathbf{8}_L$ with 3 left-handed quarks; $\mathbf{8}_R$ parity-reversed 2. $\mathbf{10}_L$ with 3 left-handed quarks; $\mathbf{10}_R$ parity-reversed 3. $\mathbf{3}_L \otimes \overline{\mathbf{3}_R}$ with 1 left-handed quark and 2 right-handed quarks; $\mathbf{3}_R \otimes \overline{\mathbf{3}_L}$ parity-reversed 4. $\mathbf{3}_L \otimes \mathbf{6}_R$ with 1 left-handed quark and 2 right-handed quarks; $\mathbf{3}_R \otimes \mathbf{6}_L$ parity-reversed

Next, anomaly matching. We assume a vector-like spectrum of baryons so that whenever a left-handed baryon appears, so too does its right-handed counterpart - this ensures the $U(1)_V$ anomaly is trivially satisfied. Here we encounter our second subtlety - since the IR theory is strongly interacting, it could be that some of these baryon species appear with the opposite handedness due to an interaction with a massless spin 1 gluon.

Luckily, it's not difficult to take this into account. We'll assign an index $p_\alpha \in \mathbf{Z}$ with $\alpha = 1, \ldots, 4$ to each of the 4 (anomaly-producing) baryons that we've listed, that captures the overall effect of gluon interactions - if the baryons are left-handed we take $p_\alpha > 0$, right-handed $p_\alpha < 0$, with the magnitude $|p_\alpha|$ giving the effective number of baryon species of that type. Our task is then to determine which (p_α) satisfy anomaly matching - or more precisely, to show that none exist!

We first consider the nonabelian $[SU(N_f)_L]^3$ anomaly, remembering that right-handed fermions contribute the negative of their anomaly coefficient, and summing over the indices of any right-handed quarks present. The anomaly expression then becomes

$$\begin{aligned} \mathcal{A} &= p_1(0-0) + p_2(27-0) + p_3(3 \times 1 - 3 \times (-1)) + p_4(6 \times 1 - 3 \times 7) \\ &= 27p_2 + 6p_3 - 15p_4 \end{aligned} \tag{3.3.4}$$

Then we check the mixed $U(1)_V \times [SU(N_f)_L]^2$ anomaly. Giving each quark charge 1 under $U(1)_V$ means each baryon has charge 3 - we may divide through by this to get a contribution proportional to the Dynkin index:

$$\begin{aligned} \frac{\mathcal{A}'}{3} &= p_1(6-0) + p_2(15-0) + p_3(3 \times 1 - 3 \times 1) + p_4(6 \times 1 - 3 \times 5) \\ &= 6p_1 + 15p_2 - 9p_4 \end{aligned} \tag{3.3.5}$$

Remembering that $N_c = 3$, we thus get the following pair of diophantine equations:

$$27p_2 + 6p_3 - 15p_4 = 3 \tag{3.3.6}$$

$$6p_1 + 15p_2 - 9p_4 = \frac{3}{3} = 1 \tag{3.3.7}$$

Immediately we see that the mixed anomaly equation cannot be satisfied, simply because the left-hand side is a multiple of 3 and the right-hand side is not! This completes the proof for $N_f = 3$ flavours (up, down, strange), guaranteeing us massless Goldstone bosons in the IR!

There are some subtleties that occur with different values of N_f, which we state here:

- QCD starts off with $N_f = 6$ flavours of quark (the previous 3 plus charm, bottom and top), so one might wonder whether the heavier quarks could make a difference to this calculation. Here we can recall our proof that the pion was the lightest particle in the spectrum, where we derived a uniform bound on the smeared fermion propagator $|S(\alpha,\beta)| \leq \frac{|\alpha||\beta|}{m} e^{-m|\beta-\alpha|}$, where m was a common bare mass given to all quarks. Applying this to the baryon propagator and looking at asymptotics we learn that $M \geq 3m$ for M the baryon mass - in other words (suitably generalised) baryons are at least as massive as their constituent quarks. So the quarks heavier than Λ_{QCD} cannot help, since we cannot approximate them as massless for t' Hooft anomaly matching.

- The baryons relevant to our world only require the up and down quark. The strange quark has $m_{\text{strange}} < \Lambda_{\text{QCD}}$ but $<<$ is a bit of a stretch. Therefore, one might wonder whether the argument carries through for $N_f = 2$ flavours. We won't give the details but it turns out that the corresponding anomaly equations *do* have solutions. This doesn't mean massless baryons actually arise, however - just that we can't use anomaly matching to rule them out. Ultimately we must turn to numerical results on the lattice which suggest no massless baryons arise in this case.

4 From Pions to Baryons

We've covered a lot of difficult mathematics so far in this essay, so it's worth pausing a little to catch our breath. Slow down, relax, be calm... in fact, it might be worth taking a short walk outside, get some fresh air. After all, the goal of physics is to help describe the world around us - it won't do any harm to appreciate it a little more.

And perhaps, on your outdoor adventure, you might find yourself walking beside a river. The water's surface shimmers brilliantly, and the current (an *actual* current, not the Noether-type we've been studying) gracefully meanders through the twists and turns of the stream's path. A cool summer breeze flows past you, producing waves that propagate gently along the direction of the wind.

Suddenly, you notice the water level starting to rise in an unusual way. And that's when you spot it - a "rounded, smooth and well-defined heap of water", large in size and progressing along the river with great speed. You find yourself turning to follow it, having to increase your pace to keep up. It continues gliding effortlessly, never seeming to diminish in size or velocity. Jogging, running, eventually breaking out into a sprint to catch up lost ground, you find it difficult to remain level with the mass before long - it's easily going 8 or 9mph. (Or maybe you're better at running than I am, and get to appreciate it for longer!)

You've just laid eyes on the "singular and beautiful phenomenon" first documented and studied by a Scotsman, John Scott Russell, some 190 years ago [1] - a soliton. Back then, Russell had a horse to help him travel along with the wave, and managed to do so for some 2 miles, before building a 30 foot wave tank in his garden to study the phenomenon more closely. And your chance encounter not only gave you a deeper appreciation for the beauty of nature - it's given you a glimpse, surprisingly enough, at the object underlying the fundamental constituents of matter. You're now well-rested and well-prepared to continue with this essay - so, read on!

4.1 Pion Physics

4.1.1 Pion Creation

We've shown that t' Hooft anomaly matching implies that our global symmetry group G_F gets broken in the IR. It's simplest to assume that the anomaly-free vector subgroup $SU(N_f)_V \times U(1)_V$ survives, and that it's the chiral part of $SU(N_f)_L \times SU(N_f)_R$ that breaks - hence, chiral symmetry breaking! We'll be following sections 5.1, 5.2 and 5.4 of [2] here.

We expect, then, to have a nonzero vacuum expectation value of some order parameter. This operator is a *chiral condensate*

$$\langle \bar{\psi}_{-i}\psi_{+j}\rangle = -\sigma\delta_{ij} \tag{4.1.1}$$

, diagonal in flavour-space. Here σ has units of [mass]^3, and so scales like Λ^3_{QCD}.

Let's check what symmetries get broken. Since the correlation function involves a quark and an anti-quark, $U(1)_V$ acts trivially, so the abelian symmetry is unbroken. But a chiral $SU(N_f)_L \times SU(N_f)_R$ rotation, which acts as

$$\psi_{-i} \to L_{ij}\psi_{-j} \text{ and } \psi_{+i} \to R_{ij}\psi_{+j} \tag{4.1.2}$$

causes the condensate to transform as

$$\langle \bar{\psi}_{-i}\psi_{+j}\rangle \to -\sigma(L^\dagger R)_{ij} \tag{4.1.3}$$

We see that this preserves the condensate if and only if $L = R$, so that overall the global symmetry group of our theory is broken as

$$G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R \to U(1)_V \times SU(N_f)_V \tag{4.1.4}$$

to its anomaly-free vector subgroup. The chiral generators then describe rotations around the $SU(N_f)$ vacuum manifold.

By Goldstone's Theorem, we're guaranteed massless fermions in the spectrum, corresponding to these broken generators - pions! Since those generators are axial, the pions will end up being pseudoscalars. Indeed, in the microscopic theory, their current is the axial

$$J^a_{A\mu} = \bar{\psi}_i T^a_{ij}\gamma_\mu\gamma^5\psi_j \tag{4.1.5}$$

[1] See http://www.ma.hw.ac.uk/~chris/scott_russell.htmlfor more details.

where T^a_{ij} are $\mathfrak{su}(N_f)$ generators (by considering the action of the axial flavour symmetry on the QCD Lagrangian).

What about in the macroscopic, low-energy theory? For this, we need to build an effective low-energy theory of these goldstone bosons. In the chiral limit with massless quarks, we simply analyse all possible lowest-order couplings and obtain the chiral Lagrangian

$$\mathcal{L}_2 = \frac{f_\pi^2}{4}\mathrm{tr}(\partial^\mu U^\dagger \partial_\mu U) \tag{4.1.6}$$

as the unique leading-order action describing pion dynamics. Here, the matrix $U = L^\dagger R \in SU(N_f)$ describes fluctuations in the value of the condensate $\langle \bar{\psi}_{-i}\psi_{+j}\rangle = -\sigma U_{ij}$, while f_π is called the *pion decay constant* - an energy scale that takes the value $f_\pi \approx 93$MeV in our world. Some authors use different conventions for the value of f_π - for example, [4] and [5] take this part of the Lagrangian with a $\frac{f_\pi^2}{16}$ factor, which corresponds to doubling $f_\pi \to 186$MeV. Another convention is to take the "geometric mean" of these giving $f_\pi \approx 130$MeV.

The pions themselves appear as the logarithm of U,

$$U(x) = \exp(\frac{2i}{f_\pi}\pi(x)) \text{ with } \pi(x) = \pi^a(x)T^a \tag{4.1.7}$$

Here $\pi(x)$ is valued in the Lie algebra $\mathfrak{su}(N_f)$, making the $\pi^a(x)$ being canonically normalised real scalar fields and T^a the normalized generators of $\mathfrak{su}(N_f)$ with $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

To make the connection with the microscopic theory, we need the analogue of $J^a_{A\mu}$. It's convenient to write this as $J^a_{L\mu} - J^a_{R\mu}$ corresponding to the currents for $SU(N_f)_L$ and $SU(N_f)_R$. For $J^a_{L\mu}$, we consider the infinitesimal transformation

$$L = e^{i\alpha^a T^a} \approx 1 + i\alpha^a T^a \tag{4.1.8}$$

The action on U is $U \to L^\dagger U$, which infinitesimally gives

$$\delta_L U = -i\alpha^a T^a U \tag{4.1.9}$$

Using the standard trick of elevating $\alpha^a \to \alpha^a(x)$, we find $\delta\mathcal{L} = \partial_\mu \alpha^a = J^a_{L\mu}$ for

$$J^a_{L\mu} = \frac{if_\pi^2}{4}\mathrm{tr}(U^\dagger T^a \partial_\mu U - (\partial_\mu U^\dagger)T^a U) \tag{4.1.10}$$

Expanding in terms of the pion fields and using $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$, we get

$$J^a_{L\mu} \approx -\frac{f_\pi}{2}\partial_\mu \pi^a \tag{4.1.11}$$

A similar calculation for $R = e^{i\alpha^a T^a} \approx 1 + i\alpha^a T^a$ with a corresponding action $U \to UR$, so $\delta_R U = i\alpha^a U T^a$, gives

$$J^a_{R\mu} = \frac{if_\pi^2}{4}\mathrm{tr}(-T^a U^\dagger \partial_\mu U + (\partial_\mu U^\dagger)UT^a) \approx +\frac{f_\pi}{2}\partial_\mu \pi^a \tag{4.1.12}$$

so that we have the formula

$$J^a_{A\mu} \approx -f_\pi \partial_\mu \pi^a \tag{4.1.13}$$

in the macroscopic theory. As expected, this creates pions from the vacuum, so that

$$\langle 0|J^a_{A\mu}(x)|\pi^b(p)\rangle = -if_\pi \delta^{ab} p_\mu e^{-ip\cdot x} \tag{4.1.14}$$

4.1.2 Pion Masses

Of course, the pions aren't truly massless in our world, since the bare masses of the quarks come into play. It's simple to take this into account, however, by treating these as a perturbation on top of the massless theory. We introduce the $N_f \times N_f$ mass matrix

$$M = \mathrm{diag}(m_1, \ldots, m_{N_f}) \tag{4.1.15}$$

where the m_i are the (suitably renormalized!) quark masses. The leading order chiral Lagrangian then becomes

$$\mathcal{L}_2 = \int d^4x \frac{f_\pi^2}{4} \mathrm{tr}(\partial^\mu U^\dagger \partial_\mu U) + \frac{\sigma}{2} \mathrm{tr}(MU + U^\dagger M^\dagger) \tag{4.1.16}$$

with σ the constant in (4.1.1) providing appropriate dimensions. Expanding in terms of pion fields then gives

$$\mathcal{L}_2 = \mathrm{tr}(\partial\pi)^2 - \frac{\sigma}{f_\pi^2} \mathrm{tr}(M + M^\dagger)\pi^2 + \ldots \tag{4.1.17}$$

giving the expected mass term.

4.1.3 Pion Decay

We've successfully constructed a low-energy effective theory for the lightest particles of QCD, viewing them as pseudo-Goldstone bosons. But interactions with other forces of nature destroy their stability.

First, we should identify where the pions of our world come from. Taking $N_f = 2$ for simplicity, the neutral π^0 and the charged $\pi^\pm$ sit in the 2×2 matrix π as

$$\pi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} \tag{4.1.18}$$

Why this arrangement? The simplest answer comes from considering how to couple the chiral Lagrangian (4.1.6) to electromagnetism - namely, we gauge a $U(1)_{\mathrm{EM}} \subset SU(3)_V$ subgroup with generator

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \tag{4.1.19}$$

corresponding to the charges of the up and down quark. This appears in our Lagrangian by replacing $\partial_\mu \to \mathcal{D}_\mu$, giving

$$S = \int d^4x \frac{f_\pi^2}{4} \mathrm{tr}(\mathcal{D}^\mu U^\dagger \mathcal{D}_\mu U) \tag{4.1.20}$$

with $\mathcal{D}_\mu U = \partial_\mu U - ieA_\mu[Q, U]$. Inspecting how $[Q, U]$ acts on π then confirms that the $\pi^\pm$ have charge $\pm e$ and the π^0 is neutral.

Next, charged pion decay. $\pi^+ = u\bar{d}$ has a lifetime of $\sim 10^{-8}$s, decaying via the weak-force as

$$\pi^+ \to \bar{\mu} + \nu_\mu$$

. We can use Fermi's four-fermion action to describe this,

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{\sqrt{2}}[\bar{u}\gamma^\mu(1-\gamma^5)d][\bar{\mu}\gamma_\mu(1-\gamma^5)\nu_\mu] \tag{4.1.21}$$

This splits the decay into two pieces - a leptonic part $\langle\bar{\mu}\nu_\mu|\bar{\mu}\gamma_\mu(1-\gamma^5)\nu_\mu|0\rangle$, which can be computed perturbatively (since it's the weak force), and a quark part $\langle 0|\bar{u}\gamma^\mu(1-\gamma^5)d|\pi^+\rangle$. But this involves the strong force which resists attempts at microscopic perturbation theory.

Here, our earlier discussion of currents saves us - the operator we need can be written as

$$\bar{u}\gamma_\mu(1-\gamma^5)d = 2(J^1_{L\mu} + iJ^2_{L\mu}) \tag{4.1.22}$$

in the microscopic theory. We then identify this with the corresponding currents in the low-energy macroscopic theory, using a modification of (4.1.13),

$$\langle 0|J^a_{L\mu}(x)|\pi^b(p)\rangle = -i\frac{f_\pi}{2}\delta^{ab}p_\mu e^{-ix\cdot p} \tag{4.1.23}$$

Finally, using $\pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2)$, we get a matrix element of

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma^5)d|\pi^+\rangle = -i\sqrt{2}f_\pi p^\mu e^{-ip\cdot x} \tag{4.1.24}$$

We see that f_π sets the scale of pion decay in our world, hence the name "pion decay constant".

The neutral pion $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$ decays even faster (lifetime $\sim 10^{-16}$s), via the electromagnetic force, as

$$\pi^0 \to \gamma\gamma$$

. In fact, this decay can be traced back to our discussion of the chiral anomaly in (2.3), and historically led to the discovery of the phenomenon - gauging our $U(1)_{\text{EM}}$ introduces an anomaly for the axial pion currents. Comparing with our result (2.3.31), we get

$$\partial^\mu J^a_{A\mu} = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\mathrm{tr}(\frac{\sigma^a}{2}Q^2) \tag{4.1.25}$$

Only the $a = 3$ component of this current is non-vanishing (the others make $\sigma^a Q^2$ have zeroes on the diagonal) , with

$$\partial^\mu J^3_{A\mu} = \frac{N_c}{96\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{4.1.26}$$

Since $J^3_{A\mu}$ is the current creating the neutral pion, we see that this "anomalous Ward identity" provides an amplitude for $\pi^0 \to \gamma\gamma$.

While the pions of our world are short-lived (at least on human timescales), all is not lost. We return to the chiral Lagrangian, and apply a trick that allows us to twist the pion fields it describes into particles topologically protected from decay - baryons!

4.2 The Missing Symmetry

It's not immediately clear why we should expect the chiral Lagrangian to be able to describe baryons - indeed, our matrix-valued field U has no charge under the $U(1)_V$ symmetry. It appears impossible to obtain a suitable conserved baryon number current in the IR!

Except this is too hasty - we're implicitly assuming that it must be a Noether current, arising from the variation of some term in a Lagrangian involving U under an infinitesimal action of the $U(1)_V$ global symmetry (which, to be clear, would always vanish since U has charge 0, describing mesons). But where else might one obtain conserved currents? If not by symmetry, how can we protect a conserved quantity from change?

It turns out the answer arises from topology - the trick is to choose something invariant under continuous deformation! Here's the argument - at any given time-slice, a field configuration of U is simply a map $U : \mathbb{R}^3 \to SU(N_f)$. But if we insist that the field asymptote to a fixed vacuum state asymptotically (to obtain a finite-energy solution), so e.g.

$$U(\mathbf{x}) \to 1 \text{ as } |\mathbf{x}| \to \infty \tag{4.2.1}$$

we're effectively working in the one-point compactification of space, S^3. So, we may think of configurations at a given time slice as being *topologically* described by a map

$$U(\mathbf{x}) : S^3 \to SU(N_f) \tag{4.2.2}$$

These are classified by their winding number

$$\Pi_3(SU(N_f)) = \mathbb{Z} \tag{4.2.3}$$

which can be computed by an integral

$$B = -\frac{1}{24\pi^2}\int d^3x \epsilon_{ijk}\text{tr}(U^\dagger(\partial_i U)U^\dagger(\partial_j U)U^\dagger\partial_k U) \tag{4.2.4}$$

using the convention that $\epsilon_{123} = 1$, matching with section 5.3 of [2] (up to a minus sign in the mostly-minus convention).

Now, this "topological charge" is quantized, and thus protected under continuous deformation. Here's the kicker - time-evolution itself is such a deformation! So we expect B to be conserved. In fact, we can write down a local current

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(U^\dagger(\partial_\nu U)U^\dagger(\partial_\rho U)U^\dagger\partial_\sigma U) \tag{4.2.5}$$

which is conserved, so $\partial_\mu B^\mu = 0$, just by virtue of the anti-symmetric ϵ tensor. This gives a corresponding conserved charge $\int d^3x B^0 = B$. (Note that lowering ϵ^{0ijk} to ϵ_{ijk} picks up a minus sign in our mostly-minus convention, which then matches with the expression before).

We've lost a $U(1)_V$ current by writing down a low-energy effective theory describing our mesonic pions, but simultaneously gained another through topological considerations. So it's very tempting to identify them and treat B as describing the baryon number of our solution. Since strong coupling makes drawing a direct link between the UV and IR theory difficult, we'll be pragmatic and simply ask whether this actually provides a good model of the proton and neutron.

4.3 The Skyrme Model

We run into a problem attempting to construct static solitons from our two-derivative chiral Lagrangian - there's no lengthscale in the game! If we start with a solution $U_\star(\mathbf{x})$ with energy $E_\star$, a rescaling of spatial coordinates $U_\lambda(\mathbf{x}) = U_\star(\lambda\mathbf{x})$ gives energy

$$E_\lambda = \frac{f_\pi^2}{4}\int d^3x \mathrm{tr}\partial_i U_\star^\dagger(\lambda\mathbf{x})\cdot\partial_i U_\star(\lambda\mathbf{x}) = \frac{1}{\lambda}E_\star \tag{4.3.1}$$

by the chain rule and changing variables in the measure. This shows that any putative static solution is unstable, preferring to stretch out over spacetime. (Note that while f_π is dimensionful, it multiplies the whole action and so doesn't help provide a lengthscale for U).

Higher-derivative terms will provide our stability. Four-derivative terms appear at the next order - using that $U^\dagger\partial_\mu U \in \mathfrak{su}(N_f)$, so traceless, together with judicious use of $0 = \partial_\mu(UU^\dagger) = (\partial_\mu U)U^\dagger + U(\partial_\mu U^\dagger)$, we find that the possible terms are

$$\mathcal{L}_4 = a_1(\mathrm{tr}\partial^\mu U^\dagger\partial_\mu U)^2 + a_2(\mathrm{tr}\partial_\mu U^\dagger\partial_\nu U)(\mathrm{tr}\partial^\mu U^\dagger\partial^\nu U) + a_3\mathrm{tr}(\partial_\mu U^\dagger\partial^\mu U\partial_\nu U^\dagger\partial^\nu U)$$

. By dimension counting the a_i are dimensionless.

The effective action should be expected to contain all such terms with specific choices of a_i. But there's a particular linear combination which suits our purposes, giving an overall Lagrangian

$$L = \int\left(\frac{f_\pi^2}{4}\mathrm{tr}(\partial^\mu U^\dagger\partial_\mu U) + \frac{1}{32g^2}\mathrm{tr}([U^\dagger\partial^\mu U, U^\dagger\partial^\nu U][U^\dagger\partial_\mu U, U^\dagger\partial_\nu U])\right)d^3x \tag{4.3.2}$$

This is called the *Skyrme* model - here g^2 is yet another dimensionless coupling constant (unrelated to the gauge coupling!) that will set the size of the soliton relative to f_π.

Admittedly this feels a little ad-hoc, but our linear combination is the unique term containing no more than two time derivatives. This is convenient for interpreting the classical equations of motion, which we'll focus on since the soliton is big and heavy. And it turns out there's a nice geometric description of the *static* energy in terms of deformation [6], that we'll now show.

To do this, we first scale out dependence on the coupling constants by writing

$$x^\mu = \frac{1}{f_\pi g} x'^\mu \tag{4.3.3}$$

We also introduce the dimensionless $\mathfrak{su}(N_f)$ current

$$L_\mu = U^\dagger \frac{\partial}{\partial x'^\mu} U \tag{4.3.4}$$

. The Lagrangian then takes the form

$$L = \frac{f_\pi}{2g} \int d^3x \left[-\frac{1}{2}\mathrm{tr}(L^\mu L_\mu) + \frac{1}{16}\mathrm{tr}([L^\mu, L^\nu][L_\mu, L_\nu]) \right] \tag{4.3.5}$$

The associated Euler-Lagrange equation is

$$\partial_\mu (L^\mu + \frac{1}{4}[L^\nu, [L_\nu, L^\mu]]) = 0 \tag{4.3.6}$$

which takes the form of a conserved current.

In this section we'll focus on finding a classical static solution for $N_f = 2$ flavours, so it'll be useful to separate out the time dependence. We split the Lagrangian as

$$L = -E_{\text{static}} + T$$

for

$$E_{\text{static}} = \frac{f_\pi}{2g} \int d^3x \left[-\frac{1}{2}\mathrm{tr}(L_i L_i) - \frac{1}{16}\mathrm{tr}([L_i, L_j][L_i, L_j]) \right] \tag{4.3.7}$$

and

$$T = \frac{f_\pi}{2g} \int d^3x \left[-\frac{1}{2}\mathrm{tr}(L_0 L_0) - \frac{1}{8}\mathrm{tr}([L_0, L_i][L_0, L_i]) \right] \tag{4.3.8}$$

the kinetic energy, where we've picked up various minus signs from lowering spatial indices. For now we discard T.

We then expand $L_i = iA_{ia}\sigma_a$ where σ_a are the Pauli matrices, and the A_{ia} are real. Using $\sigma_a \sigma_b = \delta_{ab} I_2 + i\epsilon_{abc}\sigma_c$ and the double-epsilon identity, we obtain an expression

$$\mathcal{E} = \frac{f_\pi}{2g}(\mathrm{tr}(A^T A) + \frac{1}{2}(\mathrm{tr}(A^T A))^2 - \frac{1}{2}\mathrm{tr}(A^T A A^T A)) \tag{4.3.9}$$

The combination A^TA appears frequently, and it turns out this has a nice geometric interpretation. U describes a map from $\mathbb{R}^3 \to SU(2) \cong S^3$, each of which are Riemannian manifolds with standard metrics. The partial derivatives then give us tangent vectors on S^3. But near any *fixed* $\mathbf{x}_0$, we may write $U(\mathbf{x}) = U(\mathbf{x}_0)\tilde{U}(\mathbf{x})$ where $\tilde{U}(\mathbf{x}_0) = I$. We then see that

$$L_\mu(\mathbf{x}_0) = U^\dagger(\mathbf{x}_0)U(\mathbf{x}_0)(\partial_\mu \tilde{U})(\mathbf{x}_0) = (\partial_\mu \tilde{U})(\mathbf{x}_0) \tag{4.3.10}$$

As expected, the Lie group structure allows us to restrict to the study of the tangent vectors, and metric, near the identity.

Explicitly, we may write a general element of $SU(2)$ as $U = a_0 + ia^k\sigma^k$ with a_0 and a_i real - the condition $U^\dagger U = 1$ then imposes $a_0^2 + \sum_i a_i^2 = 1$, which also guarantees that $\det(U) = 1$. This exhibits the homeomorphism $SU(2) \cong S^3$, so that the metric may be written as

$$ds^2 = da_0^2 + \sum_i da_i^2$$

. Near the identity, which corresponds to the "north pole" $(a_1, a_2, a_3, a_0) = (0, 0, 0, 1)$, the tangent space is flat, giving a metric

$$ds^2 = \sum_i da_i^2 \tag{4.3.11}$$

Locally, then, the matrix A denotes an (inverse) jacobian of the map $\mathbb{R}^3 \to S^3$, with the "strain tensor" A^TA measuring the deformation induced by such a map. We let its eigenvalues be $\lambda_1^2, \lambda_2^2, \lambda_3^2$ with the signs chosen so that

$$\lambda_1\lambda_2\lambda_3 = -\det(A) \tag{4.3.12}$$

Note that since $\det(A^TA) = \det(A)^2$ we do have $\lambda_1^2\lambda_2^2\lambda_3^2 = \det(A)^2$, but in general the λ_i are not the eigenvalues of A. (For example, if A was a 90 degree rotation it would have complex eigenvalues, but all the λ_i would be equal to 1).

In terms of this, the static energy density may be written as

$$\mathcal{E} = \frac{f_\pi}{2g}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) \tag{4.3.13}$$

Luckily for us, the baryon number density $\mathcal{B}$ (4.2.4) also takes a simple form in terms of the λ_i, again by expanding the L_i, using pauli matrix and ϵ identities:

$$\mathcal{B} = \frac{1}{2\pi^2}\lambda_1\lambda_2\lambda_3 \tag{4.3.14}$$

Summing the inequality $(|\lambda_1| - |\lambda_2||\lambda_3|)^2 \geq 0$ together with its cyclic permutations then derives a Bogomol'nyi bound on the energy of our solution:

$$\mathcal{E} \geq \frac{6\pi^2 f_\pi}{g}|\mathcal{B}| \implies E \geq \frac{6\pi^2 f_\pi}{g}|B| \tag{4.3.15}$$

where we've integrated over space in the last step.

Normally, one would express this via completing the square in $\mathcal{E}$ and solving the resulting first-order equations - indeed, these are how instantons arise via the self-dual Yang-Mills equations. It turns out that these have no solution, but the reason why is rather subtle - to saturate the bound, we need $\lambda_i = \pm\lambda_j\lambda_k$ as (i, j, k) cycle around $(1, 2, 3)$, forcing all to have modulus 1. This means the map U is an isometry, but this is impossible since $\mathbb{R}^3$ and S^3 are not isometric! (As anyone who's attempted to wrap a higher-dimensional football would know.)

We can trace this obstruction back to the failure of the one-point compactification to preserve the flat metric on $\mathbb{R}^3$ - even though we may *topologically* identify U with a map $S^3 \to S^3$, this doesn't work at the level of differential geometry. Thus, one must solve the full Euler-Lagrange equations (4.3.6), or at least their static formulation.

In terms of the A matrix, one can check these equations take the form

$$\nabla \cdot ([1 + \mathrm{tr}(A^T A)]A - AA^T A) = 0 \tag{4.3.16}$$

with the divergence acting on the leftmost (spatial) index, and that the baryon number density takes the form

$$\mathcal{B} = -\frac{2}{24\pi^2}\epsilon_{ijk}\epsilon_{abc}A_{ia}A_{jb}A_{kc} = -\frac{1}{2\pi^2}\det(A) \tag{4.3.17}$$

agreeing with (4.3.12) and (4.3.14).

4.3.1 The Classical Hedgehog

The ansatz we'll use to solve (4.3.6) is the *hedgehog*,

$$U_{\mathrm{Skyrme}}(\mathbf{x}) = \exp(if(r)\sigma \cdot \hat{\mathbf{x}}) = \cos f(r) + i\boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \sin f(r) \tag{4.3.18}$$

with σ^a denoting the triplet of Pauli matrices. The name originates from the fact that the pion fields $\pi = \sin(f(r))\hat{\mathbf{x}}$ point radially outwards from the origin everywhere - at constant radius, it looks like a "spiky ball".

There's quite a large amount of algebra to follow. Here we summarise tricks and identities the author found useful:

- $\nabla g(r) = g'(r)\hat{\mathbf{x}}$
- $\partial_i(\hat{x}_j) = \frac{\delta_{ij} - \hat{x}_i\hat{x}_j}{r}$, or compactly $\nabla(\hat{\mathbf{x}}) = \frac{I - \hat{\mathbf{x}}\hat{\mathbf{x}}}{r}$.
- $\sin^2(2f) + 4\sin^4(f) = 4\sin^2(f)$ which can be verified by expanding into complex exponentials - this combination appears surprisingly frequently.

We'll now provide a series of "checkpoints" for the interested reader to follow and compare with their own working.

Firstly, the dimensionless $\mathfrak{su}(2)$ current should be obtained as $L_i := U^\dagger\partial_i U$ $:= iA_{ia}\sigma^a$ for an A matrix

$$A_{ia} = \frac{\sin(2f)}{2r}(\delta_{ia} - \hat{x}_i\hat{x}_a) + f'\hat{x}_i\hat{x}_a + \frac{\sin^2(f)}{r}\epsilon_{iap}\hat{x}_p \tag{4.3.19}$$

There are some trace identities involving this that are useful - we quote

$$\mathrm{tr}(A^T A) = \frac{2\sin^2(f)}{r^2} + (f')^2 \tag{4.3.20}$$

and

$$\mathrm{tr}(A^T A A^T A) = \frac{2\sin^4(f)}{r^4} + (f')^4 \tag{4.3.21}$$

We also obtain that

$$\det(A) = \frac{f'\sin^2(f)}{r^2} \tag{4.3.22}$$

The static Skyrmion Euler-Lagrange equation (4.3.6) takes the form $\partial_i D_{ia} = 0$ with

$$\begin{aligned} D_{ia} = &\left[\{1 + \frac{2\sin^2(f)}{r^2} + (f')^2\}\frac{\sin(2f)}{2r} - \frac{\sin^2(f)\sin(2f)}{2r^3}\right](\delta_{ia} - \hat{x}_i\hat{x}_a) \\ &+ \left[\{1 + \frac{2\sin^2(f)}{r^2} + (f')^2\}f' - (f')^3\right]\hat{x}_i\hat{x}_a \\ &+ \left[\{1 + \frac{2\sin^2(f)}{r^2} + (f')^2\}\frac{\sin^2(f)}{r} - \frac{\sin^4(f)}{r^3}\right]\epsilon_{iap}\hat{x}_p \end{aligned} \tag{4.3.23}$$

where one may recognise the $(1 + \mathrm{tr}(A^T A))$ factor within the curly brackets.

Writing this as

$$D_{ia} = d_1(r)(\delta_{ia} - \hat{x}_i\hat{x}_a) + d_2(r)(\hat{x}_i\hat{x}_a) + d_3(r)\epsilon_{iap}\hat{x}_p \tag{4.3.24}$$

we obtain that

$$\partial_i D_{ia} = (r d_2'(r) + 2d_2(r) - 2d_1(r))\frac{\hat{x}_a}{r} \tag{4.3.25}$$

hence the required ODE to determine is

$$r d_2'(r) + 2d_2(r) - 2d_1(r) = 0$$

Substituting in the form of our coefficients and multiplying through by a factor of r then gives

$$(r^2 + 2\sin^2(f))f'' + 2rf' + \sin(2f)((f')^2 - 1 - \frac{\sin^2(f)}{r^2}) = 0 \tag{4.3.26}$$

as the ODE for f.

We also need boundary conditions on f - to have $U(\mathbf{x}) \to 1$ asymptotically we need $f(\infty) = 0$. And for U to be well-defined at the origin we need the $i\sin(f(r))\hat{\mathbf{x}}\cdot\boldsymbol{\sigma}$ term to vanish when $r = 0$, meaning

$$f(0) = \pi n \text{ for some } n \in \mathbb{Z} \tag{4.3.27}$$

Using these, we can substitute (4.3.22) into the baryon number integral (4.3.17) and obtain

$$B = -\frac{4\pi}{2\pi^2}\int_0^\infty f'\sin^2(f)dr = \frac{1}{\pi}f(0) = n \tag{4.3.28}$$

- thus, we'll set $n = 1$ for the solution supposed to model the proton/neutron.

We can use this to solve the ODE numerically and obtain the energy (thus mass) of the solution

$$\begin{aligned} E &= \frac{f_\pi}{2g}4\pi\int_0^\infty (r^2(f')^2 + 2\sin^2(f)(1+(f')^2) + \frac{\sin^4(f)}{r^2})dr \\ &= \frac{6\pi^2 f_\pi}{g}\frac{1}{3\pi}\int_0^\infty (r^2(f')^2 + 2\sin^2(f)(1+(f')^2) + \frac{\sin^4(f)}{r^2})dr \\ &= \frac{6\pi^2 f_\pi}{g}N_0 \text{ for } N_0 = 1.232 \end{aligned} \tag{4.3.29}$$

according to chapter 9 of [5]. We see that our classical hedgehog exceeds the Bogomol'nyi bound by around 23%.

4.3.2 The Quantum Hedgehog

We've found a static soliton solution to the Skyrme model with $N_f = 2$ flavours, through our hedgehog ansatz. Ideally we'd like this to describe the proton and neutron, and be able to extract their predicted masses (in the chiral limit of massless quarks). But all we know about our solution is that it has baryon number 1 - how are we going to distinguish different baryon species?

The answer lies in symmetry. We built our model to be invariant under the full chiral $SU(2)_L \times SU(2)_R$ symmetry, but the requirement that $U(\mathbf{x}) \to 1$ as $\mathbf{x} \to \infty$ breaks this to the vector subgroup $SU(2)_V$ - we can identify this with *isospin*. It acts as

$$\exp(if(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}) \to S\exp(if(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})S^\dagger \text{ with } S \in SU(2)_V \tag{4.3.30}$$

From a geometric viewpoint, this corresponds to a rotation of the output $SU(2)$ space, which doesn't affect the strain tensor defined in the last section - hence, the baryon number and energy of our solution remain unchanged.

Expanding the exponential as before and projecting to the $\boldsymbol{\sigma}$ components with $\text{tr}(\sigma^a\sigma^b) = 2\delta^{ab}$, we see that this has the effect of a spatial rotation

$$\exp(if(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}) \to \exp(if(r)\boldsymbol{\sigma}\cdot(R\hat{\mathbf{x}})) \tag{4.3.31}$$

where

$$R_{ij} = \frac{1}{2}\text{tr}(\sigma_i S\sigma_j S^\dagger) \tag{4.3.32}$$

is the $SO(3)$ element associated with S. By looking at infinitesimal actions, we find that a rotation of angle θ about axis $\hat{\mathbf{n}}$ is represented by

$$S = \exp(-\frac{i\theta}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}) \tag{4.3.33}$$

However, since $\hat{\mathbf{x}}$ sits in the argument of the Skyrmion solution, a true spatial rotation by R has the opposite effect

$$\exp(if(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}) \to \exp(if(r)\boldsymbol{\sigma}\cdot(R^{-1}\hat{\mathbf{x}})) = S^\dagger \exp(if(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})S \tag{4.3.34}$$

The upshot is that we may trade spatial rotations $SU(2)_{\text{rot}}$ for inverse isospin transformations $SU(2)_V$, with our classical hedgehog being invariant under a combined isospin-rotation.

This also allows us to parametrise the configuration space of a stationary Skyrmion - we write our general solution as

$$U(t,\mathbf{x}) = C(t)U_H(\mathbf{x})C(t)^\dagger \text{ for } C \in SU(2) \tag{4.3.35}$$

with U_H our classical hedgehog solution. Our strategy will be to write the Hamiltonian, and indeed all physical observables, in terms of our time-dependent $C(t)$ matrix. By considering how $SU(2)_{\text{rot}} \times SU(2)_V$ acts on our solution, we'll be able to assign quantum numbers for spin and isospin, and identify the proton and neutron wavefunctions! We follow the method of [4].

Onto the algebra. Like the previous section, we'll provide "checkpoints" for the interested reader to follow.

Firstly, we consider the dimensionless $\mathfrak{su}(2)$ current $L_\mu = U^\dagger\partial_\mu U$. We find that $L_i = CL_{i,H}C^\dagger$ where $L_{i,H}$ was the corresponding current for the hedgehog solution - this is because C has no spatial dependence. Since we're just conjugating by C, this doesn't change our answers for E_{static} from the previous section, so we can focus on the kinetic term.

For the temporal component, it's useful to introduce an auxiliary real vector $\mathbf{c}$ defined by

$$C^\dagger\partial_0 C = i\mathbf{c}\cdot\boldsymbol{\sigma} \tag{4.3.36}$$

This has the property that

$$\|\mathbf{c}\|^2 = \frac{1}{2}\text{tr}\left(\partial_0 C\partial_0 C^\dagger\right) \tag{4.3.37}$$

In terms of this auxiliary vector, we find that

$$L_0 = C(iB_{ia}c_i\sigma_a)C^\dagger \tag{4.3.38}$$

for a B matrix

$$B_{ia} = -2\sin^2(f)(\delta_{ia} - \hat{x}_i\hat{x}_a) + \sin(2f)\epsilon_{iap}\hat{x}_p \tag{4.3.39}$$

This allows us to express the kinetic energy as

$$T = \frac{f_\pi}{2g}\mathbf{c}^T\left[\int d^3x\left(BB^T(1+\text{tr}(A^TA)) - BA^TAB^T\right)\right]\mathbf{c} \tag{4.3.40}$$

The new terms have expressions

$$(BB^T)_{ij} = 4\sin^2(f)(\delta_{ij} - \hat{x}_i\hat{x}_j) \tag{4.3.41}$$

and

$$(BA^TAB^T)_{ij} = \frac{4\sin^4(f)}{r^2}(\delta_{ij} - \hat{x}_i\hat{x}_j) \tag{4.3.42}$$

Together with the previous expression (4.3.20), we find that the kinetic energy takes the form

$$T = \frac{8\pi f_\pi}{3g}\text{tr}(\partial_0 C\partial_0 C^\dagger)\int_0^\infty r^2\sin^2(f)(1+(f')^2+\frac{\sin^2(f)}{r^2})dr \tag{4.3.43}$$

To avoid having to evaluate this numerically, we look to Adkins', Nappi's and Witten's paper [4]. By comparing the form of their equation for their F to our (4.3.26) we see that our coordinates differ by a factor of 2, so that $f(\frac{1}{2}r) = F(r)$. Thus, we make a substitution $r = \frac{1}{2}\tilde{r}$ in our integral to find that

$$T = \frac{\pi f_\pi}{3g}\text{tr}(\partial_0 C\partial_0 C^\dagger)\Lambda \tag{4.3.44}$$

where [4] finds $\Lambda = 50.9$ numerically.

After much algebra, we've arrived at the full Lagrangian for our Skyrme model,

$$L = -E_H + \frac{\pi f_\pi}{3g}\Lambda\text{tr}(\partial_0 C\partial_0 C^\dagger) \tag{4.3.45}$$

where $E_H = \frac{6\pi^2 f_\pi}{g}N_0$ was the energy of our static classical hedgehog. We may expand $C = q_0 + i\mathbf{q}\cdot\boldsymbol{\sigma}$ with $q_0^2+\mathbf{q}^2 = 1$. In terms of these dimensionless position coordinates our Lagrangian becomes

$$L = -E_H + \frac{2\pi f_\pi}{3g}\Lambda\sum_{\alpha=0}^{3}(\dot{q}_\alpha)^2 \tag{4.3.46}$$

Normally, we'd plough ahead and define conjugate momenta, determine the classical Hamiltonian, and then quantize turning $p \to -i\hbar\nabla$. However, if one does this naively, the resulting Hamiltonian is dimensionally inconsistent! What's gone wrong?

It's a rather subtle point - back when we rescaled spacetime into dimensionless coordinates in (4.3.3), we also rescaled time as $t = \frac{1}{f_\pi g}t'$. The issue is that this rescaling of time is *not* a canonical transformation - it affects conjugate momenta without affecting our position variables q_α - and so we can't just take the Legendre transform to get the Hamiltonian.

Instead, we first undo the rescaling of time to get

$$L = -E_H + \frac{2\pi}{3g^3 f_\pi}\Lambda\sum_{\alpha=0}^{3}(\dot{q}_\alpha)^2 \tag{4.3.47}$$

We can now legally define conjugate momenta $p_\alpha = \frac{4\pi}{3g^3 f_\pi}\Lambda\dot{q}_\alpha$, giving us a Hamiltonian

$$H = p_\alpha\dot{q}_\alpha - L = E_H + \frac{3g^3 f_\pi}{8\pi\Lambda}\sum_{\alpha=0}^{3}p_\alpha^2 \tag{4.3.48}$$

We then convert these to operators in the usual way, giving us a (dimensionally consistent!) quantum Hamiltonian

$$\hat{H} = E_H + \frac{3g^3 f_\pi}{8\pi\Lambda} \sum_{\alpha=0}^{3} \left(-\frac{\partial^2}{\partial q_\alpha^2} \right) \tag{4.3.49}$$

with the constraint $\sum_{\alpha=0}^{3} q_\alpha^2 = 1$.

The constraint means we should interpret the Laplacian as really being the spherical Laplacian on S^3 - we can compute this as the ordinary Laplacian by extending the function to $\mathbb{R}^4 \backslash \{0\}$ leaving it constant on radial lines. Much like spherical harmonics, the eigenfunctions will be polynomials in the q_α, satisfying e.g.

$$-\nabla^2 (q_0 + iq_1)^l = l(l+2)(q_0 + iq_1)^l \tag{4.3.50}$$

We also want to identify spin and isospin quantum numbers, and for these we need the corresponding operators. Consider a wavefunction

$$\psi(C) = \psi(q_0, q_1, q_2, q_3)$$

. An isospin transformation acts on our solution as

$$C(t)U_H(\mathbf{x})C(t)^\dagger \to SC(t)U_H(\mathbf{x})C(t)^\dagger S^\dagger$$

for $S \in SU(2)_V$, so sends $C \to SC$. However, since C sits in the argument of our wavefunction, the corresponding action on ψ is

$$\psi(C) \to \psi(S^\dagger C) \text{ for isospin} \tag{4.3.51}$$

Analogously, a rotation acts on our solution as

$$C(t)U_H(\mathbf{x})C(t)^\dagger \to C(t)S^\dagger U_H(\mathbf{x})SC(t)^\dagger$$

, so $C \to CS^\dagger$, but on our wavefunction as

$$\psi(C) \to \psi(CS) \text{ for rotation} \tag{4.3.52}$$

To identify the corresponding operators, we use that a generator $\hat{G}$ of a symmetry transformation with parameter λ should act on a wavefunction as $\psi \to \exp(-i\lambda\hat{G})\psi$ (in units with $\hbar = 1$). Using the identification with $SO(3)$ matrices from (4.3.33), we obtain the equations

$$\exp(-i\theta\hat{n}_k I_k)\psi = \psi\left(\exp\left(\frac{i\theta}{2}\hat{n}_k\sigma_k\right)C\right) \tag{4.3.53}$$

for isospin I_k, and

$$\exp(-i\theta\hat{n}_k J_k)\psi = \psi\left(C\exp\left(-\frac{i\theta}{2}\hat{n}_k\sigma_k\right)\right) \tag{4.3.54}$$

for spin J_k. Expanding for infinitesimal θ and using Pauli matrix identities, one obtains

$$\begin{aligned} I_k &= \frac{i}{2}\left(q_0\frac{\partial}{\partial q_k} - q_k\frac{\partial}{\partial q_0} - \epsilon_{klm}q_l\frac{\partial}{\partial q_m}\right) \\ J_k &= \frac{i}{2}\left(q_k\frac{\partial}{\partial q_0} - q_0\frac{\partial}{\partial q_k} - \epsilon_{klm}q_l\frac{\partial}{\partial q_m}\right) \end{aligned} \tag{4.3.55}$$

To complete our discussion of quantization, we should discuss statistics. The classical hedgehog should be quantized as a fermion rather than a boson, and this is easy to implement - since we've identified $SO(3)$ with $SU(2)$ through our C matrix acting to rotate the solution, we simply require our wavefunctions to satisfy $\psi(-C) = -\psi(C)$. Thus, we consider only odd polynomials in the q_α.

This means that the nucleons, with total isospin and spin $I = J = \frac{1}{2}$, will correspond to wavefunctions linear in the q_α, while the deltas, with $I = J = \frac{3}{2}$, will correspond to cubic functions. The interested reader can find expressions for the resulting baryon wavefunctions in [4] - those for the proton and neutron (when spin up) are given by

$$|p\uparrow\rangle = \frac{1}{\pi}(q_1 + iq_2), |n\uparrow\rangle = \frac{i}{\pi}(q_0 + iq_3) \tag{4.3.56}$$

Finally, the masses themselves. The eigenvalues of H are given by

$$E = E_H + \frac{3g^3 f_\pi}{8\pi\Lambda}l(l+2) \tag{4.3.57}$$

where $l = 2J$ (much like spherical harmonics). This means the nucleon and delta masses are given by

$$M_N = E_H + \frac{3g^3 f_\pi}{8\pi\Lambda}(3), M_\Delta = E_H + \frac{3g^3 f_\pi}{8\pi\Lambda}(15) \tag{4.3.58}$$

Recalling $E_H = \frac{6\pi^2 f_\pi}{g}N_0$, together with the values $M_N = 938\text{MeV}$ and $M_\Delta = 1232\text{MeV}$, we obtain $g \approx 5.45$ and $f_\pi \approx 64.6\text{MeV}$ - about 30% lower than the experimental value of 93MeV!

5 Conclusion

We've finally managed to build a model describing the particles making up the vast majority of mass in our world - the proton and neutron. Now, 30% error isn't exactly anything to write home about, but even our best numerical simulations of QCD on the lattice don't get much better than around 5% accuracy starting from first principles. Frankly it's astonishing we were able to be as accurate as we were with so little external input, purely by focusing on the mathematical details of QCD.

The Skyrme model has found numerous applications beyond a simple theory of nucleons. One obvious generalization is to, quite literally, think bigger - one can add the strange quark taking $N_f = 3$, for example, to describe some of the heavier baryons, or consider solutions with higher winding numbers in an attempt to describe atomic nuclei. Taken to the extreme, one can attempt to describe the "nuclear pasta" present in neutron stars [7] as some monstrous Gordian knot.

Even beyond applications to QCD, Skyrmions have found their way into other areas of physics whenever such nonlinear sigma models appear - condensed matter physics provides a wealth of examples. For example, in the Quantum Hall effect the analog of the pion fields describe spin orientations within the Landau levels, and looking at the low-energy theory you find Skyrmions arising as excitations upon the ground state [8]. In this case the winding number also corresponds to a $U(1)$ quantum number, but not baryon number - instead, one finds that the winding counts the *electric charge* of the Skyrmion! The Zeeman effect from application of a magnetic field balances with the resulting Coulomb interaction to set the size of the soliton, and it's been suggested these charged excitations provide a route to superconductivity.

And more broadly, we've learned just how fruitful it is to be well-acquainted with the mathematical underpinnings of a physical theory. Physics is hard, and far from approximations or perturbative expansions it gets even harder. But by patiently trusting the mathematics, being guided by both symmetry and its breaking, we extracted insights and a tractable, verifiable model of the world we see around us. Nature, like a good story, loves to twist and turn!

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