Infinite Games in Set Theory: Supercompactness of $\aleph_1$ and $\aleph 2$

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1 Introduction

1.1 Outline and Motivation

The study of large cardinals has been of central interest to set theoreticians for over a century, due in part to connections between certain cardinal properties and the consistency strength of various set theoretic axioms. Large cardinal hypotheses have come to be considered a measure against which the consistency strength of such axioms could be assessed. Particular interest revolves around conjectures relating to sets of real numbers. The usefulness of the notion of large cardinals is most clearly demonstrated by their capability to furnish models of set theory as definable levels of the von Neumann hierarchy, a fact which goes hand-in-hand with the non-provability of their existence within the theory itself. Measurable cardinals are central to the class of large cardinals and have attracted much study, initially motivated by the desire to better understand Lebesgue-measurability.

Supercompactness emerged as an extension of the ideas of measurability and compactness, motivated by an exploration of reflection properties. Reflection is a curious and powerful phenomenon exhibited by cardinals with certain large cardinal properties, whereby the existence of sets with certain characteristics may be "reflected down" into lower levels of the hierarchy, giving rise to models containing sets with the said characteristics. Supercompactness also extends the scope of consistency analysis. Various hypotheses of interest in set theory were found to be consistent relative to the existence of supercompact cardinals, further motivating their study. An assumption of the existence of supercompact cardinals also simplifies the proofs of various theorems which are not strictly dependent on them.

The Axiom of Determinacy (AD) concerns the winning strategies of certain infinite games, usually played on the set of real numbers. Determinacy takes the exploration of set theory in a different direction from that of the Axiom of Choice (AC), with which it is incompatible. Choice functions over some infinite sets are still available, with a tradeoff taking place between the strength of determinacy assumed and that of choice that remains. Working in AD and its variants removes the need to consider some of the more pathological sets produced by AC, whilst still allowing choice over some infinite sets. AD may be considered to be a statement about the potential complexity of sets of real numbers, giving for example the result that every set of real numbers is Lebesgue measurable and has the perfect set property. In this context, certain properties which are normally associated only with large cardinals, (i.e. those too large to be shown to exist under ZFC), may occur in even the smallest infinite cardinals. Freed from the constraints of AC, cardinals are no longer all directly comparable in terms of size, and AD challenges the very notion of largeness. Variations on the axiom allow for different levels of determinacy and corresponding levels of partial choice. The existence of cardinals with the "large cardinal" properties that these axioms guarantee, sheds light on the consistency strengths of set theoretic axioms, as mentioned above.

In this paper we define the properties of measurability and supercompactness. In each case, these definitions can be framed in terms of elementary embeddings into a transitive model of set theory or via the existence of ultrafilters with certain properties. We demonstrate the equivalence of these two methodologies in the case of supercompactness. We then look at some of the implications of AD on the size of cardinals exibiting these properties, with particular emphasis on supercompactness results for $\aleph_1$. We conclude with an overview of some of the many other results relating to cardinal properties under the Axiom of Determinacy, and implications arising from supercompactness.

Sources for specific results are referenced throughout, with general background material from Jech [5] and Kanamori [6]. The axioms of ZF are assumed throughout. AC is not taken for granted but is required in some sections, in which cases it is clearly noted.

1.2 Notation

Most of the abbreviations used are completely standard. We will use AC and AD as shorthand for the Axioms of Choice and Determinacy respectively, ZF to refer to the axioms of Zermelo-Fraenkel set theory, and ZFC for ZF+AC. The class of ordinals will be referred to as Ord. The cardinality of a set X will be denoted by $|X|$ and its order-type by $\mathrm{ot}(X)$. We will use X^c to mean the complement of set X within a larger set but only when the context is obvious. $\langle a, b\rangle$ will refer to an ordered pair. For f a function, $f[X]$ will mean the image set of the elements of X under f. For κ a cardinal, $\mathcal{P}_\kappa(X)$ will refer to the set of subsets of X of cardinality less than κ. κ and λ will always refer to cardinals, α and β to ordinals.

Other terminology will be introduced at the time of use.

2 Background and Preliminaries

We begin with a review of some basic background material. In this section we will assume the axioms of ZFC but will draw attention to dependence on the Axiom of Choice when appropriate.

2.1 Filters and Measures

One of the central notions in the consideration of cardinal behaviour is that of a filter. A filter U over a non-empty set A is a collection of subsets of A that does not include the empty set and is closed under both supersets and finite intersections. Note it is enough to know that the intersection of any two elements of U is in U, as closure under finite intersections then follows by induction. A filter may be thought of as identifying those subsets of a set which possess a certain characteristic and separating them from those that do not, for example those subsets that are in some manner "large" relative to others. A filter may have any or none of the following characteristics:

- If $\exists X \subseteq A$ such that $\{X\} \in U$, then U is a *principal filter*;
- If $\forall X \subseteq A$ either X or its complement is in U, then U is an *ultrafilter*;
- If U is closed under intersections of any number of elements up to (but not necessarily including) a cardinal λ, then U is said to be λ*-complete*. The requirement that U be closed under finite intersections makes all filters $\aleph_0$-complete. Filters that are $\aleph_1$-complete are referred to as countably complete. We will often seek κ-completeness for a filter over a cardinal κ, which implies that if A_α is a sequence of subsets of κ indexed by some ordinal $\lambda < \kappa$, then $\bigcap_{\alpha<\lambda} A_\alpha \in U$.

A *two-valued measure* over a set A is a map, $\mu : \mathcal{P}(A) \to \{0,1\}$, mapping the empty set and singletons to zero and satisfying finite additivity of disjoint unions. The existence of a non-principal, countably-complete ultrafilter over A is easily seen to be equivalent to the existence of a non-trivial two-valued measure, with elements of the filter being those subsets of A having non-zero measure. Under this correspondence, the filter characteristic of λ-completeness coincides with that of λ-additivity in the measure. Properties that hold on sets of measure-one under measure μ may be referred to as holding on μ-large sets or μ-almost everywhere. Measures and ultrafilters will be referred to interchangeably throughout this paper, according to ease of explanation.

Note that κ-additivity is the strongest possible additivity for a measure over κ, since κ itself (which must have measure one) is comprised of κ singletons (each of measure zero). We can however consider the interaction of a larger number of filter sets if we restrict the notion of intersection to that of diagonal intersection.

Definition 2.1. *For $\{X_\alpha : \alpha < \kappa\}$ a sequence of subsets of κ, the diagonal intersection is defined as follows:*

$$\triangle_{\alpha<\kappa} X_\alpha := \{\xi < \kappa : \xi \in \bigcap_{\alpha<\zeta} X_\alpha\}.$$

Definition 2.2. *A filter over κ is said to be normal if it is closed under diagonal intersections.*

We will frequently have cause to consider ultrafilters over the set of bounded subsets of a set A. In this case the definition of a normal filter is modified as follows. In this and the rest of this section, κ is a cardinal and A a set of cardinality $\geq \kappa$.

Definition 2.3. *For $\{X_\alpha : \alpha < \kappa\}$ a sequence of subsets of $\mathcal{P}_\kappa(A)$, the diagonal intersection is defined as follows:*

$$\begin{aligned}\triangle_{\alpha<\kappa} X_\alpha &:= \{x \in \mathcal{P}_\kappa(A) : x \in \bigcap_{\alpha\in x} X_\alpha\} \\ &= \{x \in \mathcal{P}_\kappa(A) : \forall \alpha \in x,\ \ x \in X_\alpha\}.\end{aligned}$$

Definition 2.4. *A κ-complete ultrafilter U over $\mathcal{P}_\kappa(A)$ is said to be fine if:*

$$\forall x \in A, \{S \in \mathcal{P}_\kappa(A) : x \in S\} \in U.$$

So a fine filter contains as an element the collection of all the sets with x in them.

Definition 2.5. *A fine ultrafilter U over $\mathcal{P}_\kappa(A)$ is said to be a normal ultrafilter if it is closed under diagonal intersections, i.e. for X_α as above, if $X_\alpha \in U$ for all $\alpha < \kappa$, then $\triangle_{\alpha<\kappa} X_\alpha \in U$.*

Note that the definition of normality for a filter over $\mathcal{P}_\kappa(A)$ includes a requirement of κ-completeness and fineness. A normal measure is defined in the same way.

The following lemma gives an equivalent means of thinking about normal filters. First we need some definitions.

Definition 2.6. *For $X \subseteq \kappa$, a function $f : X \to \kappa$ is called a regressive function if $f(\alpha) < \alpha$ for all non-zero $\alpha \in X$. For a set A, a function $f : \mathcal{P}_\kappa(A) \to A$ is called a choice function (sometimes a push-in function) if $f(X) \in X$ for all $X \in \mathcal{P}_\kappa(A)$.*

Lemma 2.7. *A κ-complete non-principal ultrafilter U over κ is normal if and only if every regressive function on U is constant on at least one element of U.*

Proof. This was proved on Example Sheet 2 in [10]. ☐

Lemma 2.8. *A κ-complete fine ultrafilter U over $\mathcal{P}_\kappa(A)$ is normal if and only if every choice function on U is constant on at least one element of U.*

Proof. The proof is along similar lines to the above. ☐

We shall treat this as an alternative definition of normality for a filter on $\mathcal{P}_\kappa(A)$ which has already been shown to be fine, and use it without specific reference.

2.2 Measurable Cardinals

The equivalence of ultrafilters and two-valued measures motivates the naming of the following.

Definition 2.9. *A cardinal κ is said to be measurable if there exists a κ-complete non-principal ultrafilter over κ.*

Under ZFC, it is not possible to prove the existence of any measurable cardinals but, if they are assumed to exist, then they must be very large relative to known cardinals. Without the Axiom of Choice however, the situation is very different. Indeed we will see that under certain conditions incompatible with AC, not only are measurable cardinals guaranteed to exist, but cardinals as small as $\aleph_1$ can exhibit measurability.

Some properties of measurable cardinals that will be useful in the rest of this paper now follow.

Lemma 2.10. *If ultrafilter U witnesses the measurability of κ, then U includes all end-segments of κ.*

Proof. Take $\gamma < \kappa$. Then for any $\alpha < \gamma$, U a non-principal ultrafilter implies $\kappa \setminus \{\alpha\} \in U$. Hence, by κ-completeness, $\bigcap_{\alpha<\gamma}(\kappa \setminus \{\alpha\}) \in U$. But this is just $\{\alpha : \gamma \leq \alpha < \kappa\}$, as required. ☐

Lemma 2.11. *Every measurable cardinal κ admits a normal measure on κ.*

Proof. The proof was covered in Example Sheet 2 of [10]. ☐

Lemma 2.12. *Every measurable cardinal κ admits a normal measure on $\mathcal{P}_\kappa(\kappa)$.*

Proof. Given a normal measure μ on κ, for $A \subseteq \mathcal{P}_\kappa(\kappa)$ we define $\mu^*(A) := \mu(A \cap \text{Ord})$. Note that A is a collection of subsets of κ so $A \cap \text{Ord}$ is a collection of ordinals each less than κ, hence a subset of κ, so the definition makes sense. That μ^* is a κ-additive measure is trivial. For fineness we need to show that for all $x \in A$:

$$\mu^*(\{S \subseteq \kappa : x \in S\}) = 1.$$

By definition, this equals $\mu(\{S \subseteq \kappa : x \in S \text{ and } S \in Ord\}) = \mu(\{\alpha \geq x\})$, which equals one by Lemma 2.10. So μ^* is a fine measure.

Finally, let $f : \mathcal{P}_\kappa(\kappa) \to \kappa$ be a choice function on the sets of μ^*-measure one. So $f(A) \in A$ for all A such that $\mu^*(A) = 1$, i.e. such that $\mu(A \cap \text{Ord}) = 1$. Then the restriction of f to κ is a regressive function, hence by normality of μ there is a subset a of κ such that $\mu(a) = 1$ and f is constant on a. From this we can conclude that $\mu^*(a) = 1$ also and hence that μ^* is a normal measure. □

Measurability is just one characteristic amongst many forming a hierarchy of large cardinal properties, each associated with the potential but unprovable existence of a class of large cardinals possessing such properties. Members of this hierarchy can be defined in terms of a variety of different properties, leading to multiple equivalent definitions of the same class of cardinals, falling broadly into the following camps:

- Existence of measures and ultra-filters with particular properties;
- Combinatorial properties of cardinal partitions;
- Embeddings of layers of the von Neumann hierarchy into other transitive models of ZF(C);
- Properties of infinitary logic, in particular in respect of compactness.

In this section, we use the conventions and notation introduced in [10]. Here, λ will always be an inaccessible cardinal, strictly greater than κ and than α where relevant. The following theorem shows that a measurable cardinal can also be defined in terms of the existence of an elementary embedding.

Theorem 2.13. *For λ an inaccessible cardinal and $\kappa < \lambda$, κ is measurable if and only if there exists a non-trivial elementary embedding $j : V_\lambda \to M$ with critical point κ, where M is a transitive model of ZF containing $V_{\kappa+1}$.*

Proof. Given an ultrafilter U over κ, j is constructed as the canonical embedding $j : V_\lambda \to M$, where M is the Mostowski collapse of the ultrapower V_λ^κ/U.

Conversely, given an embedding j with the described characteristics, U is defined to be the set $\{X \subseteq \kappa : \kappa \in j(X)\}$, which is shown to be a κ-complete non-principal ultrafilter.

The details of the proof were covered in the Large Cardinals course [10] and are omitted here. □

2.3 Supercompact Cardinals

Embeddings can be used to define an array of cardinals further up the cardinal hierarchy. In this paper we will focus on the notion of supercompactness, defined as follows:

Definition 2.14. *For ordinal $\alpha \geq \kappa$, cardinal κ is said to be α-supercompact if there exists an elementary embedding $j : V_\lambda \to M$ with critical point κ, such that $\alpha < j(\kappa)$ and $M^\alpha \subseteq M$ (M is closed under sequences of elements of length up to α).*

Definition 2.15. *A cardinal κ is said to be supercompact if it is α-supercompact for all $\alpha \geq \kappa$.*

The following two lemmas examine supercompactness through the lens of filters.

Lemma 2.16. *If κ is α-supercompact for some ordinal $\alpha \geq \kappa$ then there exists a normal ultrafilter on $\mathcal{P}_\kappa(\alpha)$.*

Proof. Motivated by the proof of Theorem 2.13, we consider the set U defined by:

$$U := \{X \subseteq \mathcal{P}_\kappa(\alpha) : j[\alpha] \in j(X)\}$$

and seek to prove that U has the necessary characteristics. Note that in order for this to be a valid definition we need that $j[\alpha] = \{j(\beta) : \beta < \alpha\}$ is a set. We are only able to conclude this by virtue of the fact that M is closed under α-sequences, and it is in this respect that the supercompactness comes in.
Index Set For $\mathcal{P}_\kappa(\alpha)$ to serve as a suitable index set for U we must show that it is an element of U itself, i.e. that $j[\alpha] \in j(\mathcal{P}_\kappa(\alpha)) = \mathcal{P}_{j(\kappa)}(j(\alpha))$. We have that $j[\alpha] = \{j(\beta) : \beta < \alpha\}$, which is a sequence of α ordinals in M each of which is less than $j(\alpha)$. Hence $j[\alpha] \subseteq j(\alpha)$. We know also that $|j[\alpha]| = \alpha < j(\kappa)$. So $j[\alpha]$ is a subset of $j(a)$ of size less than $j(\kappa)$ and hence an element of $\mathcal{P}_{j(\kappa)}(j(\alpha))$ as required.
Filter Conditions That $\emptyset \notin U$ and that U is closed under supersets are immediate.
κ-completeness Take $\gamma < \kappa$ and let $A := (A_\beta)_{\beta<\gamma}$ be a sequence of elements of U. So $j[\alpha] \in j(A_\beta)$ for all β, and we wish to show that $j[\alpha] \in j(\bigcap_{\beta<\gamma} A_\beta)$, i.e. that $j[\alpha]$ is an element of all the elements of the sequence $j(A)$. Since $\gamma < \kappa$ we have that $j(A) = (j(A_\beta))_{\beta<\gamma}$, giving the desired result. Hence we have shown that U is a κ-complete filter.
Ultrafilter For $X \notin U$, $j[\alpha] \notin j(X)$, hence $j[\alpha] \in \mathcal{P}_\kappa(\alpha) \setminus j(X)$, which is $j(X^c)$ by elementarity, where X^c is the complement of X. Hence $X^c \in U$ and U is an ultrafilter.
Fineness For any $\beta < \alpha$, let $X_\beta := \{x \in \mathcal{P}_\kappa(\alpha) : \beta \in x\}$; we require that $X_\beta \in U$. But this holds automatically since β is an ordinal, hence X_β contains an end-segment, all of which are in U.
Normality As we have shown U to be a fine filter, we can appeal to Lemma 2.8 to prove normality. If $f : \mathcal{P}_\kappa(\alpha) \to \alpha$ is such that $f(X) \in X$ for all $X \in U$ then we have that:

$$j[\alpha] \in j\{X : f(X) \in X\} = \{j(X) : (jf)(j(X)) \in j(X)\}.$$

Hence $j[\alpha]$ satisfies $(jf)(j[\alpha]) \in j[\alpha]$, i.e. $(jf)(j[\alpha]) = j(\gamma)$ for some $\gamma < \alpha$. But this is the same as saying that $j[\alpha] \in \{j(X) : (jf)(j(X)) = j(\gamma)\} = j(\{X : f(X) = \gamma\}$, which implies that $\{X : f(X) \in \gamma\} \in U$. Hence $f(X) = \gamma$ almost everywhere, giving that U is a normal ultrafilter, as required. □

The next lemma reverses this reasoning.

Lemma 2.17 (AC)**.** *Let κ be a cardinal, α an ordinal such that $\kappa \leq \alpha$ and U a normal ultrafilter over $\mathcal{P}_\kappa(\alpha)$. Then there exists an elementary embedding $j : V_\lambda \to M$ witnessing that κ is α-supercompact.*

Proof. We proceed much as in the proof of Theorem 2.13, with j defined to be the canonical embedding of V_λ into the transitive Mostowski collapse of V_λ^κ/U, the ultrapower of V_λ modulo equivalence under U.

Firstly we note that $j(\gamma) = \gamma$ for all $\gamma < \kappa$, by induction on γ using the κ-completeness of U (as in Theorem 2.13).

Next we show that $j(\kappa) > \alpha$, which will also give us that $crit(j) = \kappa$. Let [id] be the equivalence class of the identity function. Observe that for X a subset of $\mathcal{P}_\kappa(\alpha)$, $X \in U$ iff $[id] \in j(X)$, by the definition of j. We claim that $[id] = j[\alpha]$. For all $\gamma < \alpha$, we have that $\{x : \gamma \in x\} \in U$ by the fact that U is fine, which tells us that $j(\gamma) \in [id]$ and hence that $j[\alpha] \subseteq [id]$. Conversely, if we have a function f such that $[f] \in [id]$, then $\{x : f(x) \in x\} \in U$. So f is a choice function on U and hence, by normality, f is constant at say γ on some element of U. But this just says that $\{x : f(x) = \gamma\} \in U$, i.e. that $[f] = j(\gamma)$, and hence $[id] \subseteq j[\alpha]$, which proves the claim.

Now $j[\alpha]$ may not be an ordinal, but it will clearly be well-orderable and have the same order-type as α, hence:

$$\alpha = ot(j[\alpha]) = ot([id]).$$

Note that for all $x \in \mathcal{P}_\kappa(\alpha)$, $|x| < \kappa$ by definition, so $ot(x) < \kappa$. Therefore $\{x : ot(x) < \kappa\}$ is the whole of $\mathcal{P}_\kappa(\alpha)$ and so is certainly in U. But this simply says that $ot([id]) < j(\kappa)$, and therefore that $\alpha < j(\kappa)$ as required.

It just remains to show that M is closed under α-sequences. In other words, we need to show that if we have an α-length sequence of elements of M, say $(X_\beta)_{\beta<\alpha}$, then the set $X := \{X_\beta : \beta < \alpha\}$ is itself in M. Let us consider the functions $f_\beta : \mathcal{P}_\kappa(\alpha) \to V_\lambda$ that represent the X_β in M. We seek another such function f satisfying $[f] = X = \{[f_\beta] : \beta < \alpha\}$ and claim that $f : x \mapsto \{f_\beta(x) : \beta \in x\}$ is such a function.

If $[f_\beta] \in [f]$ then $\{x : f_\beta(x) \in f(x)\} \in U$. But this set is just $\{x : \beta \in x\}$, which again is in U by fineness. So we have that $X \subseteq [f]$.

Conversely, if $[g] \in [f]$ this means $\{x : g(x) \in f(x)\} \in U$, i.e. $\{x : g(x) = f_\beta(x)\} \in U$ for some $\beta \in x$, by the definition of f. So $[g] = [f_\beta] \in X$, giving $[f] \subseteq X$.

Therefore $[f] = X$, and M is closed under α-sequences. □

The above two lemmas enable us to conclude that under ZFC, supercompactness, like measurability, can also be defined in terms of ultrafilters.

Theorem 2.18 (AC). *A cardinal κ is α-supercompact if and only if there exists a normal ultrafilter on $\mathcal{P}_\kappa(\alpha)$.*

The equivalence of normality of the ultrafilter, i.e. closure of the filter under diagonal intersections, with that of closure of the embedding under α-sequences, demonstrates the usefulness and relevance of the concept of diagonal intersection. Whilst AC is required to demonstrate the equivalence of the two definitions, we use the ultrafilter version when working without AC. It is worth noting that filters that are ultrafilters in ZF need not be so in ZFC, as the examples we shall see throughout this paper will demonstrate. AC has the capacity to generate more complicated subsets than would otherwise exist and, as a result, makes an ultrafilter harder to build.

3 Infinite Games

3.1 Infinite Games and the Axiom of Determinacy

In this section, we summarise the idea of an infinite game and describe its relevance to the study of cardinal properties.

The games under consideration are two-player, zero sum games of perfect information, with a binary win/lose outcome. The two players take alternating turns, each of which comprises the selection of an element from a set M of permitted moves. Hence at any stage during the game, the full history of play leading up to that point may be summarised by a finite sequence of elements of M and any such finite sequence represents a valid history. The game continues in this manner for an infinite number of moves resulting in an ω-length sequence of elements of M, termed a play. The set of all such possible plays, denoted M^ω, is partitioned into two subsets, the payoff sets, with one representing a win for each player. The analysis of such games is essentially the analysis of the payoff sets, that is of the elements of $\mathcal{P}(M^\omega)$, and the complexity thereof. During the course of the game, each player must make a countably infinite number of move decisions, each based on the state of play at the time. A strategy for either player may therefore be considered to be a map τ from the set of possible states of play into the set of possible moves, $\tau : M^{<\omega} \to M$.

By way of example, let us consider a game where M is the set $\{0, 1\}$. At its conclusion, the outcome of the game is an infinite string of binary digits, which we can think of as representing a real number in the interval [0,1]. The payoff set for Player I is some pre-specified subset of this interval within which they are considered to have won. It is not hard to see that if the payoff set is simply some sub-interval, then one or other player will always have a winning strategy. It is important to be aware that a "winning strategy" refers to a procedure for winning *regardless* of the actions of the other player; i.e. a single winning strategy dominates *all* possible strategies played by the opponent. A game in which one player or the other has a winning strategy is said to be determined.

In the most commonly considered game, the set M of permitted moves comprises the natural numbers and the outcome of a game is an element of the set ω^ω. The Axiom of Determinacy (AD) is the assertion that for any payoff set $A \subseteq \omega^\omega$, the game G_A is determined.

We note some very basic facts about the sizes of the sets involved in this game. The set of moves has cardinality $\aleph_0$ and the set of possible game outcomes $2^{\aleph_0}$. The outcomes are therefore equinumerous to the real numbers and although not strictly the same set are often referred to as such. The set of possible payoff sets is $\mathcal{P}(\omega^\omega)$ and hence has cardinality $2^{2^{\aleph_0}}$, whilst strategies are maps $\tau : \omega^{<\omega} \to \omega$, of which there are $2^{\aleph_0}$.

3.2 Impact of Determinacy on the Axiom of Choice and the Cardinals

For simple payoff sets such as the single sub-interval described above it is often easy to see that the resulting game is determined. The answer becomes less obvious the more complicated is the payoff set and, as such, it is perhaps unsurprising that AC is of relevance here. The first thing to observe is that it is impossible to accept both AD and AC in their entirety, as an examination of the following simple game demonstrates.

Lemma 3.1. *The Axiom of Choice implies that not all games on ω are determined, i.e.*

$$AC \implies \neg AD.$$

Proof. The idea is to construct disjoint subsets X and Y of ω^ω such that Player I can force an outcome in X against any given strategy of Player II, and Player II can similarly force a result in Y against any known strategy of Player I.

Assume AC. We observe first that the number of possible strategies for each player is $2^{\aleph_0}$, so we can index each player's strategies by $\alpha \in \mathbb{R}$. Let the Player I and Player II strategies be (σ_α) and (τ_α) respectively. By AC, these two sets can be well-ordered, so we can construct X and Y in parallel inductively as follows. Assume that we have already selected α elements for each. We select the next element of Y, y_α, to be some element of ω^ω not already in X or Y and such that y_α is the outcome of σ_α played against some strategy of Player II. We then choose x_α similarly; to be disjoint from all elements already in $X \cup Y$ and such that x_α is the outcome of τ_α played against some strategy of Player I.

In this manner we wind up with disjoint sets X and Y such that *whichever* of the strategies (σ_α) is selected by Player I, there is some strategy that Player II can use to force an outcome in Y. Similarly, Player I can force an outcome in X against any strategy of Player II. Hence neither player has a dominating strategy. □

This paper considers the impact of AD on the cardinal numbers and as a result of the above must take place in the world of ZF without AC. For the rest of the paper we assume ZF+AD. Our inability to rely on AC must impact on our thinking in a number of ways, particularly with regard to cardinality.

In ZF, two sets are defined to have the same cardinality if there exists a bijection between them, which is an equivalence relation on the class of sets. The cardinals themselves are then simply representative examples of the equivalence classes and in many respects it is unimportant which sets are chosen as these representatives. An order relation is defined by injectivity, so that $|X| \leq |Y|$ iff there exists an injection from X into Y. Reflexivity and transitivity are immediate, whilst antisymmetry follows from Cantor-Bernstein, a ZF-theorem. Hence, without relying on AC, we have a partial order. As is well known, AC is equivalent under ZF to the assertion that all sets can be well-ordered; i.e. are in bijection with some ordinal. This enables the cardinal of a set to be defined in ZFC as the least ordinal with which it is in bijection. The resulting cardinals, a subset of the ordinals, then form a well-ordered class, which enables their enumeration as aleph-numbers indexed by the ordinals.

Without AC the situation is much more complex. Whilst the ordinals remain the same as under AC, we can no longer rely on every set having a well-ordering and hence cannot refer to the order-type of a set without first establishing that it has one. The ordering on cardinality is defined by injectivity as before but we no longer have cardinal comparability for all sets. And we must find a different set to serve as the representative cardinal for an equivalence class, since there may be no comparable ordinal. We cannot use the equivalence class itself since this is too large to be a set. But we can work around this problem using "Scott's trick"; defining the cardinal for set X to be the set of all sets in bijection with X and of minimal rank. This set may or may not be well-orderable; if it is, then it will have cardinality equal to one of the aleph numbers.

The complexity arises owing to the fact that our ordering on the cardinals is now only a partial order. Furthermore, the size comparison between two cardinals is much more subtle. If there is an injective function from set X to set Y then by definition $|X| \leq |Y|$. Under AC, the existence of such an injection is synonymous with that of a surjection from Y onto X, and the two can be thought of as equivalent. Without AC this is no longer the case. Whilst, given an injection from $X \to Y$, it is easy to construct a surjection from Y to X, the converse no longer holds. Given a surjective function, there may be a large amount of choice required to select the necessary elements for a map in the other direction.

Hence there is a variety of means for comparing the size of a set X to the ordinal numbers. If there is an injection from X into some ordinal (equivalently a surjection from an ordinal onto X), then X itself must be well-orderable and its cardinality equal to that of an aleph-number lying amongst the ordinals. If this is not the case, then we can still examine which ordinals, α, inject into X, which by definition means $\alpha \leq |X|$. The least ordinal λ for which this does *not* apply is an object of interest, the Hartog number for X, but we cannot say that $|X| \leq \lambda$, since by assumption no such injection exists. Similarly, but distinctly, we can look at the ordinals onto which X surjects, but there is no guarantee that this will be the same set as provided by the injections. Again, we can define λ^* to be the least ordinal onto which there is no surjection from X, called its Lindenbaum number. The comments above indicate that $\lambda \leq \lambda^*$ and that there is no order relationship between X and λ^*.

These ordinals are of particular interest when X is the set of real numbers. It is a consequence of AD that there is no injection from $\aleph_1$ into $\mathbb{R}$, whilst there clearly is from $\aleph_0$. Hence for the reals, $\lambda = \aleph_1$. The value of λ^* for the reals is referred to as Θ.

$$\Theta := min\{\alpha \in \text{Ord} : \nexists \text{ a surjection from } \mathbb{R} \to \alpha\}.$$

As this cannot be a successor ordinal, we may write it equivalently as:

$$\Theta = sup\{\alpha \in \text{Ord} : \exists \text{ a surjection} : \mathbb{R} \to \alpha\}.$$

Under AC, it is clear that $\Theta = (2^{\aleph_0})^+$ and with the benefit of the Generalised Continuum Hypothesis we can be even more precise and say that $\Theta = \aleph_2$. But under AD, numerous results have established that Θ is very large, including for example that it must be an aleph-fixed point. [6]

Working within the realm of AD does not however require us to reject AC in its entirety. Each of AC and AD may be considered to come in a variety of strengths, based on the sets to which they refer. The version of AD which we have described demands determinacy in games with a set of permitted moves of size ω. (We could reasonably refer to this as AD_ω.) In a more general setting, moves could be selected from any set M, and the relevant Axiom of Determinacy, AD_M, again asserts that this game is determined for all possible payoff sets $A \in \mathcal{M}^\omega$. The results we want will sometimes require us to consider games where moves can be selected from the set of real numbers, the determinacy of which we refer to as $\text{AD}_\mathbb{R}$.

AC too comes in different flavours. The fullest standard version insists on the existence of a choice function for any set of non-empty sets of any size. We can choose to weaken the conditions of AC by restricting its applicability either to fewer sets (fewer choices) or to smaller ones (simpler choices). We will use $\text{AC}_X(Y)$ to denote the Axiom of Choice as applied to $|X|$-many subsets of the set $|Y|$, noting that the axiom is dependent only on the cardinality of X and Y. For example the familiar Axiom of Countable Choice asserts the existence of a choice function for any countable collection of non-empty sets (of unspecified size), and could be denoted by $\text{AC}_\omega(\infty)$. It is clear that $AC_\kappa(\lambda) \implies AC_{\kappa'}(\lambda')$ whenever $\kappa' \leq \kappa$ and $\lambda' \leq \lambda$. Of course if X is finite the axiom is so weak as to not need stating, since one always has the ability to make a finite number of choices explicitly within the logical language.

By somewhat weakening the conditions of AC, the conflict with AD can be removed. In fact we can go further than this and see that AD actually implies a certain amount of choice.

Lemma 3.2. *AD_M implies the validity of $AC_M(M^\omega)$ for any set M.*

Proof. [9] Consider the game whereby the players are provided with a set of non-empty subsets of M^ω, itself indexed by M; $\{S_x \subseteq M^\omega : x \in M\}$. Player I begins by selecting $x_0 \in M$; the players then play an infinite game, with Player II winning if his sequence of moves lies in the set S_{x_0}. A strategy for Player II consists precisely of a choice function on non-empty subsets of M^ω, requiring M such choices. The axiom AD_M asserts that this game is determined and, as Player I clearly has no winning strategy, this implies that such a choice function must exist, i.e. that $\mathrm{AC}_M(M^\omega)$ holds. □

Hence in the basic game where $M = \omega$ we get $AD \implies AC_\omega(\mathbb{R})$, whilst for uncountable M the result simplifies to $AD_M \implies AC_M(M)$.

4 Compactness Results for $\aleph_1$

4.1 Measurability of $\aleph_1$ and $\aleph_2$

In the 1960s, $\aleph_1$ and $\aleph_2$ were first shown to be measurable by Robert Solovay, who demonstrated that under AD the club filter over $\aleph_1$ is an ultrafilter. Measurability was also established for $\aleph_2$. We state these results here without proof as a starting point for our examination of the stronger large cardinal properties of $\aleph_1$ and $\aleph_2$ under AD.

Theorem 4.1. *$AD \vDash \aleph_1$ and $\aleph_2$ are measurable cardinals.*

Proof. This result can be found as Theorem 28.2 in Kanamori [6] and is the subject of another Part III essay. □

4.2 α-supercompactness of $\aleph_1$ under $\mathrm{AD}_\mathbb{R}$

Solovay took this result further by looking at supercompactness, as outlined in [8], referencing Proceedings of the Cabal Seminar in 1976-77. In order to do this he was forced to make a stronger base assumption, namely that of determinacy over the real numbers, $\mathrm{AD}_\mathbb{R}$. With this assumption he was able to establish the α-supercompactness of $\aleph_1$ for all values of α below Θ, as follows.

In order to extend the measurability result for $\aleph_1$ to one of supercompactness, we need to demonstrate the existence of a filter over $\mathcal{P}_{\aleph_1}(\alpha)$ for some $\alpha \geq \aleph_1$ which, in addition to satisfying the conditions of measurability, is also both fine and closed under diagonal intersections. To this end, we consider the following game. The permitted moves are any finite subsets of the real numbers and the deemed outcome of the game is the union of the chosen moves, hence a countably infinite subset of $\mathbb{R}$. We define Ω to be the set of countable subsets of $\mathbb{R}$, that is $\mathcal{P}_{\aleph_1}(\mathbb{R})$. Following Solovay's convention, we define G_A in this case to be the game where A is the payoff set for Player II, noting that A is a subset of Ω. We further define the set $U := \{A \subseteq \Omega : \text{Player II wins}\}$ and explore the characteristics of U. We note that the set of finite subsets of $\mathbb{R}$ has cardinality $|\bigcup_{n<\omega}\mathbb{R}^n|$ which equals $|\mathbb{R}|$, so the set of permitted moves can be identified with $\mathbb{R}$ and therefore $\mathrm{AD}_{\mathbb{R}}$ applies to this game. We observe also that the sizes of the sets of possible outcomes, payoff sets and strategies are $|\mathbb{R}|$, $2^{|\mathbb{R}|}$ and $2^{|\mathbb{R}|}$, respectively.

We remark that if $A \in U$, Player II can still win even if he only pays attention to the game periodically. For example, he can make a random move on every odd-numbered play, and then "catch up" on the even-numbered rounds by taking the union of the three moves since his last even-numbered play and treating this as a single move by Player I. We will take advantage of this possibility in the proof which follows.

Lemma 4.2. *U is an ultrafilter on $\mathcal{P}_{\aleph_1}(\mathbb{R})$.*

Proof. We note first that for $A \in U$, A is a collection of countable subsets of $\mathbb{R}$ and hence U is of the right set type. We require to show that $\emptyset \notin U$, that U is closed under supersets and finite intersections, and that for all $A \subseteq \Omega$, either A or its complement is in U.

If $A = \emptyset$ then Player I can guarantee himself a win on his first move by playing any non-empty set, hence $\emptyset \notin U$.

If $A \in U$ this means that Player II has a strategy that guarantees him a win. If B is a superset of A, then Player II can still win by following the same strategy as he would for A, so B is also in U.

Now assume that the payoff set is the intersection of two sets, A and B, each of which is in U. This means that Player II has a winning strategy for each of G_A and G_B, say τ_A and τ_B. By definition, τ_A guarantees a win for Player II in the game G_A, regardless of the strategy pursued by Player I. By the remark above, Player II can ensure the outcome of the game lies in set A solely by making decisions on his odd-numbered moves. Similarly, the even-numbered moves can be selected to ensure an outcome in set B, demonstrating that the intersection of A and B also lies in U. (Note that it is not possible for Player II to have winning strategies for two non-overlapping payoff sets.) From this we can conclude that the set U is closed under finite intersections, and is therefore a filter.

Assume now that $A \notin U$; we show that $\Omega \setminus A$ is in U, making U an ultrafilter. Since $A \notin U$, Player II does not have a winning strategy for G_A. Hence, by determinacy, Player I does have, which is to say that Player I can force the game outcome into the set $\Omega \setminus A$. Player II can "borrow" this strategy to guarantee a win in the game $G_{\Omega\setminus A}$ as follows. Player II ignores Player I's opening move and makes his first move as if he were Player I in the game G_A, which is to say, seeking an outcome in $\Omega \setminus A$. Then on his second move, he reacts as if Player I had made only a single move, comprising the union of the two he has in fact made. He then continues to play as if he were Player I in the game G_A, thereby guaranteeing an outcome in the set $\Omega \setminus A$. Hence $\Omega \setminus A \in U$, enabling us to conclude that U is an ultrafilter. □

In order to look at countable intersections of payoff sets, we first have to tackle an issue with choice. Faced with a countable collection of elements of U, Player II has to be able to choose a winning strategy for each, and the number of possible strategies from which he can choose is $2^{2^{\aleph_0}}$; too large to be covered by $\mathrm{AC}_{\mathbb{R}}(\mathbb{R})$. So it is necessary to show that such a sequence of winning strategies can indeed be selected, which is the purpose of the following lemma.

Lemma 4.3. *Let $\langle A_n\rangle_{n\in\omega}$ be a sequence of elements of U. Then there exists a sequence of strategies $\langle\tau_n\rangle_{n\in\omega}$ such that each τ_n is a winning strategy for Player II in the game G_{A_n}.*

Proof. We consider an auxiliary game with rules as follows. First, Player I picks an integer and Player II picks the empty set, then they play the game G_{A_n} where n is the number picked by Player I. There cannot be a winning strategy for Player I (since there is not for any G_{A_n}), yet as usual the game must be determined. Hence there must be a winning strategy for Player II, and this "meta-strategy" comprises the desired choice function for strategies over G_{A_n} with $n \in \omega$. □

Armed with the selected strategies, we can now tackle countable intersections.

Lemma 4.4. *U is closed under countable intersections.*

Proof. Assume that the payoff set for Player II is $A = \bigcap_{n \in \omega} A_n$, where each $A_n \in U$.

Define a bijection $f : \omega \to \omega \times \omega$ by induction as follows:

$$f(0) := \langle 0, 0 \rangle$$

and for $f(n) = \langle a, b \rangle$,

$$f(n+1) := \begin{cases} \langle a+1, b-1 \rangle & \text{if } b \neq 0 \\ \langle 0, a+1 \rangle & \text{if } b = 0 \end{cases}$$

This gives a set of order pairs $\langle n, m \rangle$ which is increasing in each of n and m individually. Given the winning strategy for each game G_{A_n} chosen above, Player II follows a similar strategy to the one for finite intersections, only this time seeking to win all the games simultaneously, as follows. For given n, he will pursue the objective of winning game G_{A_n}, using the set of moves coded by the pair $\langle n, m \rangle$ as m ranges over the natural numbers. In so doing, he will treat all the moves by both players since the last in this sequence as if they were a single move by Player I. The key point here is that there is room in the ω moves for each objective to be assigned an infinite number of moves and, by this means, he can ensure that the outcome lies in A_n, for each $n \in \omega$.

Hence we have that U is countably-complete. □

Lemma 4.5. *U is a fine filter.*

Proof. Let $x \in \mathbb{R}$, and A be the set of all countable subsets of $\mathbb{R}$ containing x. Then Player II can guarantee an easy win, simply by picking a set containing x on his first move. The game is now guaranteed to have an outcome in one of the elements of A. Hence $\{S \in \Omega : x \in S\} \in U$, and as this is true for all $x \in \mathbb{R}$, we have that U is a fine filter. □

Lemma 4.6. *U is a normal ultrafilter.*

Proof. We need to show that U is closed under diagonal intersections. Recall that U is a set of countable subsets of $\mathbb{R}$ and therefore can be indexed by $\mathbb{R}$. Referring to Definition 2.3 above, for (A_α) a set of elements of U indexed by $\alpha \in \mathbb{R}$:

$$\triangle_{\alpha \in \mathbb{R}} A_\alpha := \{x \in \Omega : \forall \alpha \in x,\ x \in A_\alpha\}.$$

We would like to be able to show that this diagonal intersection is also an element of U. At first sight, this appears to be a much harder proposition than for countable additivity, as the intersecting sets are now indexed over the reals. However, at any particular stage, we only have to intersect $|x|$ of them, and since each x is a countable set, we have only a countable number of intersections to consider. Hence Player II can still ensure that the outcome lies inside this countable intersection in exactly the same manner as in the previous proof, namely by interlacing his optimal moves for each of the A_α. Unlike in the proof of countable completeness, the set of target payoff sets now changes with each move but because it remains countable, it will always be an achievable objective.

Once again, we have to deal with the issue of the choice involved in selecting the strategies. This is tackled via an auxiliary game in much the same way as in Lemma 4.3. Player I picks a real number x, Player II the empty set, then they play the game G_{A_x}. By $\mathrm{AD}_\mathbb{R}$ the game is determined and Player I cannot have a winning strategy so Player II must have. This strategy provides the necessary choice function over strategies for G_{A_x} as x ranges over $\mathbb{R}$. □

We are now able to conclude the desired result:

Theorem 4.7. *$AD_\mathbb{R} \vDash \aleph_1$ is α-supercompact for all $\alpha < \Theta$.*

Proof. From the five lemmas above, we have shown that U is a normal ultrafilter over $\mathcal{P}_{\aleph_1}(\mathbb{R})$. Noting that α-supercompactness is defined for ordinals α, we seek to use this to find a normal ultrafilter over some $\mathcal{P}_{\aleph_1}(\alpha)$. Letting f be a surjection that maps $\mathbb{R}$ onto some ordinal α, we define V:

$$V := \{X \subseteq \mathcal{P}_{\aleph_1}(\alpha) : \{x \in \Omega : f[x] \in X\} \in U\}.$$

That V is a countably-additive ultrafilter over $\mathcal{P}_{\aleph_1}(\alpha)$ follows directly from the definition and the corresponding characteristics of U. Fineness and normality are easily shown using the corresponding features of U as applied to the pre-images of elements and subsets of α, which can be relied upon to exist by the fact that f is a surjection. Hence we have α-supercompactness for any α onto which $\mathbb{R}$ surjects and, by definition, these are precisely the ordinals less than Θ. □

4.3 α-supercompactness of $\aleph_1$ under AD

The above gives a good result on α-supercompactness of $\aleph_1$ under the assumption of $\mathrm{AD}_\mathbb{R}$ and we would like to know whether the condition can be loosened to that of AD. In the early 1970s, Donald Martin showed that some supercompactness results were possible on the basis of AD alone, albeit for smaller values of α, as summarised in this paper by Di Prisco and Henle. [1]

Note first we already have the existence of a normal measure over $\mathcal{P}_{\aleph_1}(\aleph_1)$ by Lemma 2.12 and Theorem 4.1. Let μ_1 be such a measure. To demonstrate any level of supercompactness beyond that implied by measurability, we need to show the existence of a normal measure on countable subsets of ordinals larger than $\aleph_1$. In order to achieve this, we first push the result forward to ordinals α lying strictly between $\aleph_1$ and $\aleph_2$.

Since any such α has a cardinality of $\aleph_1$, we can define a bijection:

$$f : \alpha \to \aleph_1 \tag{4.1}$$

This bijection in turn induces a bijection f^* on the respective powersets by mapping a set to its image under f:

$$f^* : \mathcal{P}_{\aleph_1}(\alpha) \to \mathcal{P}_{\aleph_1}(\aleph_1), \text{ by } f^* : x \mapsto f[x]$$

For α such that $\aleph_1 < \alpha < \aleph_2$, we define the measure μ_α over collections of subsets of α to be the μ_1-measure of their image under f^*. Hence, for $A \subseteq \mathcal{P}_{\aleph_1}(\alpha)$,

$$\mu_\alpha(A) := \mu_1(f^*[A]).$$

Lemma 4.8. *The measure μ_α as described is a normal measure on $\mathcal{P}_{\aleph_1}(\alpha)$.*

Proof. If A is the empty set or a singleton then $f^*[A]$ is the same, hence has measure zero.

Let A be a countable union of disjoint sets, $A = \bigcup_{n\in\omega} A_n$. We have:

$$\mu_\alpha(A) := \mu_1(f^*[\bigcup_{n\in\omega} A_n]) = \mu_1(\bigcup_{n\in\omega} f^*[A_n]),$$

since the A_n are disjoint and f^* is a bijection. By the countable additivity of μ_1, this equals:

$$\sum_{n\in\omega} \mu_1(f^*[A_n]) = \sum_{n\in\omega} \mu_\alpha(A_n)$$

so μ_α is countably additive.

Referencing Definition 2.4 as applied to a measure, to demonstrate fineness we require that for all $x \in \alpha$, $\mu_\alpha(B) = 1$, where $B := \{\beta \in \mathcal{P}_{\aleph_1}(\alpha) : x \in \beta\}$. By definition, we have that:

$$\mu_\alpha(B) = \mu_1(f^*[B]) = \mu_1(\{f^*(\beta) : x \in \beta\}) = \mu_1(\{f[\beta] : x \in \beta\}) = 1$$

by the fineness of μ_1. So μ_α is a fine filter.

Finally, to demonstrate normality, let $g : \mathcal{P}_{\aleph_1}(\alpha) \to \alpha$ be a choice function on subsets of α. Then it is easy to see that $h := f \circ g \circ {f^*}^{-1}$ is a regressive function on $\mathcal{P}_{\aleph_1}(\aleph_1)$ (since f and f^* are bijections) and hence, by the normality of μ_1, there is some $X \subseteq \mathcal{P}_{\aleph_1}(\aleph_1)$ of measure one such that h is constant on X. So $\mu_1(X) = 1$ and, recalling that f^* is a bijection, we may write $X = f^*[A]$ for some $A \subseteq \mathcal{P}_{\aleph_1}(\alpha)$, giving that $\mu_\alpha(A) = 1$ and $f \circ g$ is constant on A. Therefore g is constant on A, and A serves as witness to the normality of μ_α. □

We would like to extend the α-supercompactness result further to $\alpha = \aleph_2$ itself, but of course can no longer make use of a bijection with $\aleph_1$. Instead, if we are given a specific μ_α for each α between $\aleph_1$ and $\aleph_2$, we can use a measure on $\aleph_2$ to stitch them together as follows. Let μ_2 be a normal measure on $\aleph_2$, the existence of which is guaranteed by Lemma 2.11 and Theorem 4.1. (Note, in contrast to μ_1, μ_2 is acting on the cardinal itself not its powerset.) For $A \subseteq \mathcal{P}_{\aleph_1}(\aleph_2)$ and fixed $\alpha \in (\aleph_1, \aleph_2)$, we first define $A_\alpha := A \cap \mathcal{P}_{\aleph_1}(\alpha)$. Observe that an element of A is a subset of $\aleph_2$ and an element of A_α is a subset of α, which is less than $\aleph_2$. So A_α is essentially a collection of initial segments of A. We then define:

$$\mu^*(A) := \mu_2(\{\alpha \in \aleph_2 : \mu_\alpha(A_\alpha) = 1\}). \tag{4.2}$$

Recalling that the μ's all take values 0 and 1, μ^* simply assesses whether or not the positive μ_α's form a μ_2-large set.

Once again, a potential obstacle arises from the fact that we need to exercise an $\aleph_2$-number of choices in order to make the selection of measures for the α's, all in the absence of AC. This problem is circumvented by the following lemma.

Lemma 4.9. *With the notation above, the measure μ_α is independent of the function f chosen in Equation 4.1.*

Proof. Let f and g be bijections taking $\alpha \to \aleph_1$, f^* and g^* the induced powerset bijections, and μ_f, μ_g the associated versions of μ_α. We wish to prove that $\mu_f(A) = \mu_g(A)$ for all $A \subseteq \mathcal{P}_{\aleph_1}(\alpha)$, which is accomplished by showing that $\mu_1(f^*[A]) = 1$ implies $\mu_1(g^*[A]) = 1$. Letting $h : \mathcal{P}_{\aleph_1}(\aleph_1) \to \mathcal{P}_{\aleph_1}(\aleph_1)$ be the function $g^*(f^*)^{-1}$, this is equivalent to showing that h preserves μ_1-largeness.

We define B to be the subset of $\mathcal{P}_{\aleph_1}(\aleph_1)$ on which h acts as the identity, and claim it is sufficient to show that B is μ_1-large. Indeed, if this is the case, then for any μ_1-large set A, $A \cap B$ is also large, as are each of the following sets:

$$A \cap B = h[A \cap B] \subseteq h[A]$$

which gives the desired result.

So it just remains to show that $\mu_1(B) = 1$. We will use f and g to build two choice functions on $\mathcal{P}_{\aleph_1}(\aleph_1)$ as follows:

$$j(X) := \begin{cases} \min x \in X, & \text{if } (f^*)^{-1}(X) \subseteq (g^*)^{-1}(X) \\ \min x \in X \text{ such that } f^{-1}(x) \notin (g^*)^{-1}(X) & \text{otherwise} \end{cases}$$

$$k(X) := \begin{cases} \min x \in X, & \text{if } (g^*)^{-1}(X) \subseteq (f^*)^{-1}(X) \\ \min x \in X \text{ such that } g^{-1}(x) \notin (f^*)^{-1}(X) & \text{otherwise} \end{cases}$$

Since μ_1 is normal, we can find μ_1-large subsets C and D on which j and k are constant at say γ and δ respectively, each a countable ordinal. Hence $C \cap D$ is also μ_1-large, so any element (a subset of $\aleph_1$) can be extended to include the elements $u := g(f^{-1}(\gamma))$ and $v := f(g^{-1}(\delta))$, whilst remaining in $C \cap D$. Call this extended set X and note that $j(X) = \gamma$ and $k(X) = \delta$.

Now, $\gamma \in X$ and $f^{-1}(\gamma) = g^{-1}(u) \in (g^*)^{-1}(X)$, so γ fails the second condition for determination of $j(X)$, which must therefore be determined by the first condition. Hence $(f^*)^{-1}(X) \subseteq (g^*)^{-1}(X)$. By a symmetric argument on δ we can show that $(g^*)^{-1}(X) \subseteq (f^*)^{-1}(X)$, giving us that $(g^*)^{-1}(X) = (f^*)^{-1}(X)$. Therefore $h(X) = g^*(f^*)^{-1}(X) = X$, i.e. $X \in B$. So B contains a set of measure one and we can conclude that $\mu_1(B) = 1$ as required. □

Armed with the required μ_α's, we now proceed to show that μ^* has the desired characteristics.

Lemma 4.10. *The measure μ^* described in Equation 4.2 is a normal measure on $\mathcal{P}_{\aleph_1}(\aleph_2)$.*

Proof. If A is the empty set or a singleton, then $\mu_\alpha(A_\alpha) = 0$ for all α, hence $\mu^*(A) = 0$.

For countable additivity it suffices to show that on any partition of $\mathcal{P}_{\aleph_1}(\aleph_2)$ into a countable number of subsets, exactly one is of measure one. Let $\{A_\beta\}$ be such a partition with $\beta < \delta < \aleph_1$. We note that the additivity of μ_α implies that for each α there is a unique β such that $\mu_\alpha(A_\beta \cap \mathcal{P}_{\aleph_1}(\alpha)) = 1$; let

$r : \aleph_2 \to \aleph_1$ be the map that takes α to β. Then $r^{-1}[\{\beta\}]$ is a subset of $\aleph_2$ and the values of β induce a countable partition of $\aleph_2$ in this way. The additivity of μ_2 guarantees that exactly one such subset will have measure one, giving the additivity of μ^*, as required.

For fineness, we require to show that for all $x \in \aleph_2$, $\mu^*(B) = 1$, where $B := \{\beta \in \mathcal{P}_{\aleph_1}(\aleph_2) : x \in \beta\}$. If we take $\alpha > x$, then $B_\alpha := B \cap \mathcal{P}_{\aleph_1}(\alpha) = \{\beta \in \mathcal{P}_{\aleph_1}(\alpha) : x \in \beta\}$, hence $\mu_\alpha(B_\alpha) = 1$ by fineness of μ_α. And we know that $\mu_2(\alpha : \alpha > x) = 1$ by Lemma 2.10. Hence we have that μ_2 is fine.

Now let $f : \mathcal{P}_{\aleph_1}(\aleph_2) \to \aleph_2$ be any choice function. To show that μ^* is normal, we need to find a set $A \subseteq \mathcal{P}_{\aleph_1}(\aleph_2)$ such that $\mu^*(A) = 1$ and f is constant on A. We will use the normality of both μ_2 and the μ_α's to do this.

The restriction of f to $\mathcal{P}_{\aleph_1}(\alpha)$ is also a choice function and hence by the normality of μ_α there exists some $X_\alpha \subseteq \mathcal{P}_{\aleph_1}(\alpha)$ such that $\mu_\alpha(X_\alpha) = 1$ and f is constant on X_α, that is $f[X_\alpha] = \{\beta_\alpha\}$ for some $\beta_\alpha < \alpha$.

If we now define the function $g : \aleph_2 \to \aleph_2$ by $g : \alpha \mapsto \beta_\alpha$, then g is another choice function, this time on $\aleph_2$. So we can apply the normality of μ_2 to derive a μ_2-large set $B \subseteq \aleph_2$ on which g is constant, at say β. We then consider the set $A := f^{-1}[\{\beta\}]$. f is constant on A by definition, so we have only to show that A has measure one under μ^*.

By definition, for each $\alpha \in B$, $g(\alpha) = \beta$, hence $f : X_\alpha \mapsto \{\beta\}$, giving that $X_\alpha \subseteq A \cap \mathcal{P}_{\aleph_1}(\alpha)$, which we defined to be A_α. But $\mu_\alpha(X_\alpha) = 1$, hence so does $\mu_\alpha(A_\alpha)$ for all $\alpha \in B$, a set of μ_2-measure one.

Putting all this together, we have that $\mu^*(A) := \mu_2(C)$ where $C = \{\alpha : \mu_\alpha(A_\alpha) = 1\}) \supseteq B$ and hence $\mu^*(A) = 1$ as required. □

We are now able to conclude the following:

Theorem 4.11. *AD* $\vDash \aleph_1$ *is* $\aleph_2$*-supercompact.*

5 Outline of Further Results

In this section we summarise, in outline only, some other results relating to supercompactness under AD.

We first introduce the notion of α-compactness. α-compactness is related to a compactness property of an infinitary logical language and, as with other cardinal properties, can be defined in a variety of ways which are equivalent under ZFC. The ultrafilter version of the definition is as follows.

Definition 5.1. *A cardinal κ is said to be α-compact if and only if there exists a fine ultrafilter on $\mathcal{P}_\kappa(\alpha)$.*

From this definition and Theorem 2.18 it is clear that supercompactness is simply a special case of compactness, requiring the additional constraint of normality on the filter. Solovay's result that $\mathrm{AD}_{\mathbb{R}}$ gives α-supercompactness of $\aleph_1$ for all $\alpha < \Theta$ (our Section 4.2), was predated by the following, requiring AD alone ([6], citing Kunen).

Theorem 5.2. *AD* $\vDash \aleph_1$ *is α-compact for all $\alpha < \Theta$.*

The proof proceeds by showing that any countably-complete filter over α can be extended to a countably-complete ultrafilter, an equivalent definition of α-compactness. The Solovay result can then be seen as an extension to this theorem, with normality achieved at the cost of a stronger determinacy assumption.

In Section 4.3 we used the argument presented by Di Prisco and Henle in [1] to demonstrate specifically the $\aleph_2$-supercompactness of $\aleph_1$. Their results were actually broader than this, and included the following.

Lemma 5.3. *If κ is λ-supercompact, then κ is α-supercompact for all $\lambda < \alpha < \lambda^+$.*

Lemma 5.4 (AC)**.** *Given cardinals κ and λ with λ measurable, if κ is α-supercompact for all $\kappa < \alpha < \lambda$, then κ is λ-supercompact.*

The proofs of these two lemmas are essentially the same as those presented in Section 4.3. There, the reliance on AC for the second was eliminated by the demonstration that the representative α-filters derived in the first were unique in the case considered. Extensions of these results to larger alephs are impeded by the fact that multiple filters arise, making the elimination of the dependence on AC impossible by this method. However, a minor modification to the proof enabled Di Prisco and Henle to draw the following conclusion, making use of the result that under AD, $cf(\aleph_3) = \aleph_2$.

Theorem 5.5. *$AD \vDash \aleph_2$ is $\aleph_3$-compact.*

A much broader array of results can be achieved if the focus is shifted to the projective sets, a flavour of which follows. Becker in [2], citing results of Solovay, Harrington and Kechris, assumes AD along with the Axiom of Dependent Choice (DC). Becker is dismissive of AD, going so far as to declare it "false", owing to its inconsistency with AC. By working in $L[\mathbb{R}]$, the class of all sets Gödel-constructible from the reals, this discomfort is smoothed over. Since AC implies that $L[\mathbb{R}] \vDash DC$ and AD gives $L[\mathbb{R}] \vDash AD$, $L[\mathbb{R}]$ is a natural model in which to work and explore the consequences of AD+DC. In this context we have the following.

Theorem 5.6. *$AD + DC \vDash \aleph_1$ is α-supercompact for all projective ordinals α.*

The short proof has similarities to that of Theorem 4.7. Firstly the idea of ordinal codes is introduced; each projective ordinal is encoded by a real number. Countable sets of projective ordinals can then be encoded similarly. It is then a simple matter to translate a game played on sequences of projective ordinals below λ, with payoff set a subset of λ^ω, into the classic game with payoff set a subset of ω^ω. Hence AD gives us the determinacy of these ordinal games. A specific such game is then examined, with the set of winnable payoff sets for one player shown to generate a normal ultrafilter.

In a second paper [3], Becker proves the following. Here, δ is a projective ordinal analogue of Θ.

Theorem 5.7. *$AD + DC \vDash \aleph_2$ is δ-supercompact.*

The proof again uses ordinal codes to represent projective ordinals as reals, but with additional conditions that endow sets of countable ordinals with a natural topology. A filter over $\mathcal{P}_{\aleph_2}(\delta)$ is constructed and shown to have the desired characteristics.

In a more recent paper [4], Becker and Jackson broaden these results significantly by proving the following.

Theorem 5.8. *AD + DC* $\models$ *Every projective ordinal is* α*-supercompact for all* $\alpha < \aleph_{\aleph_1}$.

We note that the first two projective ordinals are just $\aleph_1$ and $\aleph_2$.

It would be reasonable to ask what, if any, useful conclusions can be drawn from supercompactness results. The following, due to Ikegami and Trang [7], provides a beautifully simple example.

Theorem 5.9. *ZF + (*$\aleph_1$ *is supercompact)* $\models$ *DC.*

Proof. (Outline) For R a total relation on a set A, we wish to find a function $f : \omega \to A$ such that $\langle f(n), f(n+1)\rangle \in R$ for all $n \in \omega$. We use the supercompactness of $\aleph_1$ to demonstrate the existence of a countable subset σ of A satisfying the following:

$$(\forall x \in \sigma)(\exists y \in \sigma)\langle x, y\rangle \in R.$$

Since σ is countable, there is a surjection from ω onto σ, meaning we can always select an element from any subgroup. Hence, starting with any element of σ, we can remain within σ and trace out a sequence of length ω, as desired. $\square$

The trick here is that whilst A cannot be assumed to be well-orderable, the supercompactness of $\aleph_1$ enables us to find a subset that is and that still carries the total relation property.

6 Conclusions

In this essay we have looked at cardinal properties that under ZFC are associated only with inaccessible cardinals, but that in the absence of AC can manifest on much smaller cardinals. We have described infinite games, which serve as a tool for considering sets of real numbers and give rise to the Axiom of Determinacy as an alternative paradigm to that of full choice. Primary results have focused on the α-supercompactness of $\aleph_1$ for various ordinals α under different determinacy assumptions. The section on further results demonstrates the significant range of directions in which a longer essay could proceed.

The treatment has of necessity been brief and attempted only an overview of a very small corner of the world of cardinal properties under AD. There are many more published results which could form a part of a lengthier document and several directions this could take. There remain many open questions in the subject, for example in descriptive set theory, on which we touched only very briefly via the projective sets. We have not touched at all on the subject of consistency results. This is perhaps the most obvious omission, given these could be considered to be one of the primary motivations for the study of large cardinal properties, and the most natural direction for an extension of this essay.

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